

# FINDING EIGENVALUES OF HOLOMORPHIC FREDHOLM OPERATOR PENCILS USING BOUNDARY VALUE PROBLEMS AND CONTOUR INTEGRALS

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**ABSTRACT.** Investigating the stability of nonlinear waves often leads to linear or nonlinear eigenvalue problems for differential operators on unbounded domains.

In this paper we propose to detect and approximate the point spectra of such operators (and the associated eigenfunctions) via contour integrals of solutions to resolvent equations. The approach is based on Keldysh' theorem and extends a recent method for matrices depending analytically on the eigenvalue parameter. We show that errors are well-controlled under very general assumptions when the resolvent equations are solved via boundary value problems on finite domains. Two applications are presented: an analytical study of Schrödinger operators on the real line as well as on bounded intervals and a numerical study of the FitzHugh-Nagumo system.

We also relate the contour method to the well-known Evans function and show that our approach provides an alternative to evaluating and computing its zeroes.

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## 1. INTRODUCTION

Studies of the analytic Evans function ([AGJ, PW]) have found numerous applications to the stability problem for nonlinear waves. The function has proved to be an invaluable tool for locating point spectra of differential operators that are defined on the real line or, more generally, on cylindrical domains. In particular, analyzing the behavior of the Evans function near zero or near infinity provides one major step in studying point spectra rigorously. We refer to [Sa], [KKS], [GLM],[OS], [DL] for a variety of applications and extensions of the concept to other settings.

There have been quite a few approaches to computing the Evans function numerically and then determine from its zeros the unknown eigenvalues, see [Br], [BDG], [BZ], [HZ], [HSZ], [MN], [LMT], [SE]. The standard definition of the Evans function (see [Sa]) involves a determinant of vectors which depend analytically on the eigenvalue parameter and which are determined as initial values of exponentially decaying solutions on both semi-axes. This leads to the problem of integrating stiff ODEs while keeping the analyticity with respect to the parameter. For higher dimensions the evaluation of a determinant may also lead to instabilities unless a proper scaling is employed, see for instance [GLZ, Thm.4.15] for a discussion of the appropriate scaling of the Evans function. The techniques proposed to solve these problems utilize exterior products [Br],[BZ],[AB], [BDG] or solve Kato's matrix differential equation [K, Sec.II.4.2], see [HSZ].

In this paper we pursue an alternative road that avoids the intermediate stage of computing the Evans function. Rather we propose to solve the original analytical eigenvalue problem in the whole space via well-posed boundary value problems on finite domains while preserving analyticity. For the latter problem we propose a numerical method using contour integrals of solutions to resolvent equations. The approach is based on Keldysh' theorem and extends a recent method [B] for nonlinear eigenvalue problems with matrices. Our goal is to determine all eigenvalues inside a given contour  $\Gamma \subset \mathbb{C}$  and to guarantee that, at each stage of approximation, approximate equations are as well-conditioned as the original eigenvalue problem. The method applies to general nonlinear eigenvalue problems

$$F(\lambda)v = 0, \quad \lambda \in \Omega, \quad (1.1)$$

where  $F(\lambda)$  are Fredholm operators of index 0 that depend analytically on  $\lambda \in \Omega$ . For certain right-hand sides  $v$  and functionals  $w$  we require to evaluate the integrals

$$\frac{1}{2\pi i} \int_{\Gamma} \langle w, F(\lambda)^{-1}v \rangle d\lambda \quad \text{and} \quad \frac{1}{2\pi i} \int_{\Gamma} \lambda \langle w, F(\lambda)^{-1}v \rangle d\lambda, \quad (1.2)$$

where  $\Gamma$  lies in the resolvent set. The idea first appeared for the matrix case in the papers [AK08],[AK09] where it was used in connection with the Smith normal form. In [B] we arrived independently at similar expressions (without applying the functional  $w$ ) by using the theorem of Keldysh, see [MM, Ch.1]. It turns out that the Keldysh approach allows to locate eigenvalues precisely, to handle multiplicities and to simultaneously compute (generalized) eigenvectors, see Section 2.3. Moreover, Keldysh' Theorem works for abstract Fredholm operators [MM] and thus allows to generalize the whole approach as we will show in Section 2. Note that contour integrals are normally used for computing the winding number and thus the number of zeros inside the contour (see [Br], [BDG], [BZ] in case of the Evans function), whereas our method allows to locate all zeros inside the contour and, in addition, to obtain approximate eigenfunctions (Theorems 2.4, 2.8 and equation (2.23)).

In Section 3 we discuss suitable normalizations of the Evans function and relate our contour method to such a normalized Evans function (Theorems 3.13, 3.18). Then we apply the method to first order differential systems with  $\lambda$ -dependent matrices. We continue this in Section 4 and show that isolated eigenvalues are well approximated when the resolvent equations are solved on bounded intervals with suitable boundary conditions (Theorems 4.7, 4.11). Here we follow [BR] where it is shown that the functional analysis of discretization methods [V76], [V80] applies to this situation. In Section 5 we apply the theory to Schrödinger operators on the real line and express the various quantities of our approach in terms of Jost solutions and Levinson's Theorem (Theorems 5.3, 5.4). Section 6 concludes with several numerical experiments for the FitzHugh-Nagumo system which demonstrate the robustness of the contour method. In particular, we show how computational errors depend on the length of the finite interval, the number of quadrature points used for (1.2), and on the rank test used to determine the number of eigenvalues inside the contour.

## 2. ABSTRACT RESULTS FOR THE CONTOUR METHOD

**2.1. The abstract setting.** We consider nonlinear eigenvalue problems as in (1.1) where  $F : \Omega \rightarrow \mathcal{L}(\mathcal{H}, \mathcal{K})$  is a holomorphic function on a domain  $\Omega \subseteq \mathbb{C}$  with values in the space  $\mathcal{L}(\mathcal{H}, \mathcal{K})$  of bounded linear operators from some complex Banach space  $\mathcal{H}$  into another Banach space  $\mathcal{K}$ . We will assume that the operators  $F(\lambda)$  are Fredholm of index 0 for all  $\lambda \in \Omega$  (this will be written as  $F \in \mathbb{H}(\Omega, \mathcal{F}(\mathcal{H}, \mathcal{K}))$ ). As usual, cf. [MM, Ch.I], we define the resolvent set and the spectrum of the operator pencil  $F$  by

$$\rho(F) = \{\lambda \in \Omega : F(\lambda) \text{ invertible}\}, \quad \sigma(F) = \Omega \setminus \rho(F).$$

Throughout, we impose the following assumptions.

**Hypothesis 2.1.** *Assume that  $F \in \mathbb{H}(\Omega, \mathcal{F}(\mathcal{H}, \mathcal{K}))$  and  $\rho(F) \neq \emptyset$ .*

Under these assumptions  $\sigma(F)$  is a discrete subset of  $\Omega$  and the operator valued function  $F^{-1}(\cdot)$  is meromorphic, see [MM, Thm.1.3.1]. In the terminology of [GGK, Ch.IX] the function  $F^{-1}(\cdot)$  is finitely meromorphic. In general, we follow the setting in [MM, Ch.I]. In particular, we use dual spaces  $\mathcal{H}'$ ,  $\mathcal{K}'$  and denote the dual pairing by elements  $w \in \mathcal{H}'$ ,  $u \in \mathcal{K}'$  in two equivalent ways

$$w^\top v = \langle w, v \rangle, \quad v \in \mathcal{H}, \quad u^\top v = \langle u, v \rangle, \quad v \in \mathcal{K}. \quad (2.1)$$

Thus, the dual space is the space of linear (versus complex conjugate linear) functionals. As noted in [MM, Sec.1.1] it is important to avoid the complex adjoint space and complex conjugate functionals (even if  $\mathcal{H}$  and  $\mathcal{K}$  are Hilbert spaces), since this will turn holomorphic functions into antiholomorphic ones. We prefer the notion  $w^\top$  over  $w^*$  and  $w'$  because it is in accordance with matrix-vector notation. Correspondingly, we denote the dual of an operator  $A \in \mathcal{L}(\mathcal{H}, \mathcal{K})$  by  $A^\top \in \mathcal{L}(\mathcal{K}', \mathcal{H}')$ . It is defined by

$$\langle A^\top w, v \rangle = (A^\top w)^\top v = w^\top (Av) = \langle w, Av \rangle, \quad w \in \mathcal{K}', v \in \mathcal{H}.$$

Any two elements  $v \in \mathcal{H}$  and  $w \in \mathcal{K}'$  define an operator  $vw^\top \in \mathcal{L}(\mathcal{K}, \mathcal{H})$  as follows

$$(vw^\top)u := v(w^\top u) = v \langle w, u \rangle, \quad u \in \mathcal{K}. \quad (2.2)$$

Because of this definition we omit brackets and simply write  $vw^\top u$ . Note that the operator  $vw^\top$  is of rank one if  $v \neq 0, w \neq 0$ . We denote by  $\mathcal{N}(\cdot)$  the null-space and by  $\mathcal{R}(\cdot)$  the range of a linear operator.

Let  $\Gamma$  be a smooth contour in  $\Omega$  surrounding a bounded subdomain  $\text{int}(\Gamma) = \Omega_0 \subseteq \Omega$ . Since  $\Omega_0 \cup \Gamma$  is compact and eigenvalues are isolated [MM, Thm.1.3.1] there are at most finitely many eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_\varkappa \in \Omega_0$ , and, in addition, they are of finite multiplicity. Our goal is to determine these eigenvalues and good approximations of the eigenvectors by computing contour integrals of the type (1.2)

In the following choose  $m$  linearly independent functionals  $\hat{w}_j \in \mathcal{H}', j = 1, \dots, m$ , and  $\ell$  linearly independent vectors  $\hat{v}_k \in \mathcal{K}, k = 1, \dots, \ell$ . Let us assume that  $F(\lambda)$  is invertible for all  $\lambda \in \Gamma$ . Then we can solve the following equations for unknown vectors  $y_k(\lambda) \in \mathcal{H}$ :

$$F(\lambda)y_k(\lambda) = \hat{v}_k, \quad \lambda \in \Gamma, \quad k = 1, \dots, \ell. \quad (2.3)$$

We introduce an  $(m \times \ell)$ -matrix valued function,  $E(\cdot)$ , on  $\Gamma$  as follows:

$$E(\lambda) = \left( \langle \hat{w}_j, y_k(\lambda) \rangle \right)_{j,k=1}^{m,\ell}, \quad \lambda \in \Gamma. \quad (2.4)$$

In addition, we introduce the following  $(m \times \ell)$  matrices:

$$D_0 = \frac{1}{2\pi i} \int_{\Gamma} E(\lambda) d\lambda, \quad D_1 = \frac{1}{2\pi i} \int_{\Gamma} \lambda E(\lambda) d\lambda. \quad (2.5)$$

*Remark 2.2.* We note that equation (2.3) is the only information used to obtain  $E(\lambda)$  and thus the matrices  $D_0$  and  $D_1$ . So, replacing (2.3) by an approximate equation with good stability properties, the matrices  $D_0, D_1$  can be computed using approximation arguments. In our applications, equation (2.3) is a differential equation on the line, while the respective approximation is a differential equation on a finite segment with appropriate boundary conditions. We will discuss the errors of this approximation in Sections 2.4 and 4 below. In addition, there are errors caused by approximating the contour integrals (2.5) by a quadrature rule, e.g. the trapezoid sum. This error depends on the number of quadrature points and on the distance of eigenvalues to the contour. For a detailed analysis in case of analytic contours  $\Gamma$  we refer to [B]. In any rate, we may assume in what follows that the matrices  $E(\lambda), D_0, D_1$  are known to us.  $\diamond$

*Remark 2.3.* At first sight it seems unnecessarily general to have different dimensions  $m$  and  $\ell$  that may also differ from  $\varkappa$ . However, it is important for practical computations. First of all,  $\varkappa$  is generally unknown and thus a quantity to be determined. Our approach will only require  $m, \ell \geq \varkappa$  which can be achieved by increasing  $m$  and  $\ell$ . Second, the number  $\ell$  determines the number of equations (2.3) to be solved and hence the numerical effort. Therefore we want to keep it as small as possible. Finally,  $m$  can be very large since it gives the number of functionals that we can evaluate on the solutions of (2.3). In principle we can assume to know  $y_j(\lambda)$  exactly and hence all values  $w^\top y_j(\lambda), w \in \mathcal{K}'$ . In this case we may think of  $E(\lambda)$  being a matrix with infinitely long columns, see Example 4.6.  $\diamond$

**2.2. Simple eigenvalues inside the contour.** In this subsection, in addition to Hypothesis 2.1, we assume that  $F$  has only simple eigenvalues inside the contour  $\Gamma$ . That is, for some  $\varkappa \geq 0$ ,

$$\sigma(F) \cap \Omega_0 = \{\lambda_1, \dots, \lambda_\varkappa\}, \quad (2.6)$$

$\dim \mathcal{N}(F(\lambda_n)) = 1$ , and there are eigenvectors  $v_n \in \mathcal{H}$  of the operator  $F(\lambda_n)$ , the eigenvectors  $w_n \in \mathcal{K}'$  of the dual operator  $F(\lambda_n)'$  such that, cf. [MM, Def.1.7.1],

$$F(\lambda_k)v_n = 0, \quad w_n^\top F(\lambda_n) = 0, \quad w_n^\top F'(\lambda_n)v_n \neq 0, \quad n = 1, \dots, \varkappa. \quad (2.7)$$

In the following it will be convenient to normalize  $v_n, w_n$  such that

$$w_n^\top F'(\lambda_n)v_n = 1, \quad n = 1, \dots, \varkappa. \quad (2.8)$$

We recall the well-known Keldysh formula, see [MM, Thm.1.6.5]:

$$F(\lambda)^{-1} = \sum_{n=1}^{\varkappa} \frac{1}{\lambda - \lambda_n} v_n w_n^\top + H(\lambda), \quad \lambda \in \Omega_0 \setminus \{\lambda_1, \dots, \lambda_\varkappa\}, \quad (2.9)$$

where the normalization (2.8) is assumed and  $H(\cdot)$  is a holomorphic function on a neighborhood  $\mathcal{U}$  of  $\Omega_0$  with values in  $\mathcal{L}(\mathcal{K}, \mathcal{H})$ . We also use the notation from (2.2).

Using (2.9) in (2.4) yields for  $j = 1, \dots, m$ ,  $k = 1, \dots, \ell$  the following formula for the entries of the matrix  $E(\lambda)$ :

$$E_{jk}(\lambda) = \sum_{n=1}^{\varkappa} \frac{1}{\lambda - \lambda_n} \langle \widehat{w}_j, v_n \rangle \langle w_n, \widehat{v}_k \rangle + \langle \widehat{w}_j, H(\lambda) \widehat{v}_k(\lambda) \rangle. \quad (2.10)$$

We write this in matrix form by introducing the rectangular matrices  $G_l \in \mathbb{C}^{m, \varkappa}$  and  $G_r \in \mathbb{C}^{\ell, \varkappa}$ ,

$$G_l = \left( \langle \widehat{w}_j, v_n \rangle \right)_{j=1, n=1}^{m, \varkappa}, \quad G_r = \left( \langle w_n, \widehat{v}_k \rangle \right)_{k=1, n=1}^{\ell, \varkappa}, \quad (2.11)$$

the diagonal matrix  $\Lambda = \text{diag}\{\lambda_1, \dots, \lambda_\varkappa\}$ , and the matrix function

$$H_0(\lambda) = (\langle \widehat{w}_j, H(\lambda) \widehat{v}_k(\lambda) \rangle)_{j=1, k=1}^{m, \ell}.$$

Then (2.10) has the matrix form

$$E(\lambda) = G_l(\lambda I_\varkappa - \Lambda)^{-1} G_r^\top + H_0(\lambda), \quad \lambda \in \Omega, \quad (2.12)$$

where  $H_0 \in \mathbb{H}(\mathcal{U}, \mathbb{C}^{m, \ell})$ . Using Cauchy's Theorem, we evaluate the matrix  $D_0$  from (2.5) as follows:

$$D_0 = \frac{1}{2\pi i} \int_{\Gamma} G_l(\lambda I_\varkappa - \Lambda)^{-1} G_r^\top d\lambda = G_l G_r^\top. \quad (2.13)$$

A similar calculation yields

$$D_1 = G_l \Lambda G_r^\top. \quad (2.14)$$

Our standing assumption in the following will be that both matrices  $G_l$  and  $G_r$  have rank  $\varkappa$ . As we noted in Remark 2.3 this requires to have  $\ell \geq \varkappa$  and  $m \geq \varkappa$ . Equation (2.13) then implies the following formula for the number of eigenvalues  $\lambda_k$  enclosed by  $\Gamma$ :

$$\varkappa = \text{rank } D_0. \quad (2.15)$$

Thus, cf. Remark 2.2, one can compute the number  $\varkappa$  from a rank test of  $D_0$  and this should be sufficiently robust to approximation arguments.

Next, we show how  $D_1$  can be used to evaluate the actual location of the eigenvalues  $\lambda_n$  enclosed by  $\Gamma$ . And this procedure should be robust to approximation arguments as well.

Let  $\sigma_1, \dots, \sigma_{\varkappa}$  denote the nonzero singular values of the matrix  $D_0$ , and introduce the diagonal  $(\varkappa \times \varkappa)$  matrix  $\Sigma_0 = \text{diag}\{\sigma_1, \dots, \sigma_{\varkappa}\}$ . We use the short form of the singular value decomposition of  $D_0$  (e.g. [A05, §3.2])

$$D_0 = V_0 \Sigma_0 W_0^*, \quad V_0 \in \mathbb{C}^{m, \varkappa}, \quad V_0^* V_0 = I_{\varkappa}, \quad W_0 \in \mathbb{C}^{\ell, \varkappa}, \quad W_0^* W_0 = I_{\varkappa}. \quad (2.16)$$

Note that here one uses adjoint matrices  $W_0^* = \overline{W_0}^\top$  and  $V_0^* = \overline{V_0}^\top$ .

Due to equation (2.13), we have

$$D_0 = V_0 \Sigma_0 W_0^* = G_l G_r^\top. \quad (2.17)$$

Since the columns of the matrices  $G_l$  and  $V_0$  span the same subspace, there is a nonsingular  $(\varkappa \times \varkappa)$  matrix  $S$  such that  $V_0 S = G_l$ . From this we obtain

$$\Sigma_0 W_0^* = S G_r^\top \quad \text{and} \quad G_r^\top = S^{-1} \Sigma_0 W_0^*.$$

Using this in equation (2.14) we find

$$D_1 = V_0 S \Lambda S^{-1} \Sigma_0 W_0^*.$$

Multiplying the last equation by  $V_0^*$  from the left and by  $W_0 \Sigma_0^{-1}$  from the right, yields

$$S \Lambda S^{-1} = V_0^* D_1 W_0 \Sigma_0^{-1} =: D, \quad (2.18)$$

which, in turn, implies the desired formula for the  $\lambda_k$ 's:

$$\{\lambda_1, \dots, \lambda_{\varkappa}\} = \sigma(D). \quad (2.19)$$

We stress again that the spectrum of the matrix  $D$  in (2.18), by (2.4), (2.5), can be evaluated using only the data from (2.3), and thus can be obtained using approximation arguments. We summarize our result in the following theorem.

**Theorem 2.4.** *Suppose that the operator pencil  $F \in \mathbb{H}(\Omega, \mathcal{F}(\mathcal{H}, \mathcal{K}))$  has only simple eigenvalues  $\lambda_1, \dots, \lambda_{\varkappa}$  inside a simple closed contour  $\Gamma$  in  $\Omega$  and no eigenvalues on the contour. Given a set of linearly independent functionals  $\hat{w}_j \in \mathcal{H}'$ ,  $j = 1, \dots, m$ , and vectors  $\hat{v}_k \in \mathcal{K}$ ,  $k = 1, \dots, \ell$ , with  $m, \ell \geq \varkappa$ , let the matrices  $D_0, D_1 \in \mathbb{C}^{m \times \ell}$  be determined by (2.3), (2.4), (2.5) and assume that the matrices  $G_l, G_r$  from (2.11) have maximum rank. Then the  $\varkappa \times \varkappa$ -matrix  $D = V_0^* D_1 W_0 \Sigma_0^{-1}$ , where  $V_0, \Sigma_0, W_0$  are given by the singular value decomposition (2.16) of  $D_0$ , has  $\lambda_1, \dots, \lambda_{\varkappa}$  as simple eigenvalues.*

As soon as  $\Lambda$  is determined, one can view (2.18), that is, the equation

$$DS = \Lambda S, \quad (2.20)$$

as an equation for the corresponding to  $\lambda_n$  eigenvectors of the matrix  $D$ , which are the columns of the matrix  $S$ . With  $S$  known we can compute  $G_l = V_0 S$ , i.e. the values  $\langle \hat{w}_j, v_n \rangle$  are available. From this we determine good approximations of eigenvectors  $v_n$ ,  $n = 1, \dots, \varkappa$ , as follows:

**Step 1:** Select vectors  $\hat{u}_k \in \mathcal{H}$ ,  $k = 1, \dots, m$ , that are biorthogonal to the  $\hat{w}_j$ 's,

$$\langle \hat{w}_j, \hat{u}_k \rangle = \delta_{jk}, \quad j, k = 1, \dots, m. \quad (2.21)$$

**Step 2:** Determine coefficients  $\beta_{k,n}$ ,  $k = 1, \dots, m$ , which minimize

$$\Phi(\beta) = \sum_{j=1}^m |\langle \hat{w}_j, v_n - \sum_{k=1}^m \beta_{k,n} \hat{u}_k \rangle|^2. \quad (2.22)$$

Due to (2.21) the minimum is attained at  $\beta_{k,n} = \langle \hat{w}_k, v_n \rangle = (G_l)_{k,n}$ .

**Step 3:** Determine approximate eigenvectors of the operator  $F(\lambda_n)$  from

$$v_n^{\text{approx}} = \sum_{k=1}^m (G_l)_{k,n} \hat{u}_k, \quad n = 1, \dots, \varkappa. \quad (2.23)$$

If  $\mathcal{H}$  is a Hilbert space then we can identify  $\hat{w}_j = \hat{u}_j, j = 1, \dots, m$ , so that (2.21) requires these vectors to form an orthonormal system. Introducing the subspace  $X_m = \text{span}\{\hat{u}_1, \dots, \hat{u}_m\}$  we find that  $\Phi(\beta)$  in (2.22) agrees with  $\|v_n - \sum_{k=1}^m \beta_{k,n} \hat{u}_k\|^2$  up to a constant. Hence  $v_n^{\text{approx}}$  is the best approximation of  $v_n$  in the subspace  $X_m$ . If the functions  $\hat{u}_j$  are not orthonormal, one can find the best approximation of  $v_n$  in  $X_m$  by first solving for  $\beta_{j,n}$  the linear system

$$\sum_{j=1}^m \langle \hat{u}_k, \hat{u}_j \rangle_{\mathcal{H}} \beta_{j,n} = \langle \hat{w}_k, v_n \rangle_{\mathcal{H}}, \quad k = 1, \dots, m, \quad n = 1, \dots, \varkappa,$$

and then setting

$$v_n^{\text{approx}} = \sum_{j=1}^m \beta_{j,n} \hat{u}_j, \quad n = 1, \dots, \varkappa. \quad (2.24)$$

**2.3. Multiple eigenvalues inside the contour.** In this section we show that multiple eigenvalues do not produce serious problems with the contour method. In Theorem 2.8 we prove that the matrix  $D$  from (2.18) inherits the multiplicity structure of the original nonlinear problem. This is analogous to the behavior of the Evans function (see [AGJ]), see Section 3.1 for more details. Let us first recall the definition of multiplicity of eigenvalues for nonlinear pencils and the associated notion of chains of eigenvectors (see [MM, Sec.I.1.6], [V76]).

**Definition 2.5.** Let  $F \in \mathbb{H}(\Omega, \mathcal{F}(\mathcal{H}, \mathcal{K}))$  and  $\lambda_0 \in \sigma(F)$ .

(i) A tuple  $(v_0, \dots, v_{n-1}) \in \mathcal{H}^n, n \geq 1$ , is called a chain of generalized eigenvectors (CGE) of  $F$  at  $\lambda_0$  if the polynomial  $v(\lambda) = \sum_{j=0}^{n-1} (\lambda - \lambda_0)^j v_j$  satisfies

$$(Fv)^{(j)}(\lambda_0) = 0, \quad j = 0, \dots, n-1.$$

The order of the chain is the index  $r_0 (\geq n)$  satisfying

$$(Fv)^{(j)}(\lambda_0) = 0, \quad j = 0, \dots, r_0 - 1, \quad (Fv)^{(r_0)}(\lambda_0) \neq 0.$$

The rank  $r(v_0)$  of a vector  $v_0 \in \mathcal{N}(F(\lambda)), v_0 \neq 0$ , is the maximum order of CGEs starting at  $v_0$ .

(ii) A canonical system of generalized eigenvectors (CSGE) of  $F$  at  $\lambda_0$  is a system of vectors

$$v_{j,p} \in \mathcal{H}, \quad j = 0, \dots, \mu_p - 1, \quad p = 1, \dots, q, \quad q \geq 1,$$

with the following properties:

- (1)  $v_{0,1}, \dots, v_{0,q}$  form a basis of  $\mathcal{N}(F(\lambda_0))$ ,
- (2) the tuple  $(v_{0,p}, \dots, v_{\mu_p-1,p})$  is a CGE of  $F$  at  $\lambda_0$  for  $p = 1, \dots, q$ ,
- (3) for  $p = 1, \dots, q$  the indices  $\mu_p$  satisfy
$$\mu_p = \max\{r(v_0) : v_0 \in \mathcal{N}(F(\lambda_0)) \setminus \text{span}\{v_{0,\nu} : 1 \leq \nu < p\}\}.$$

(iii) The numbers  $\mu_p, p = 1, \dots, q$ , are called the partial multiplicities, where  $\mu_1 + \dots + \mu_q$  is the algebraic multiplicity and  $q = \dim \mathcal{N}(F(\lambda_0))$  is the geometric multiplicity. The subspace  $\text{span}\{v_{j,p} : j = 0, \dots, \mu_p - 1, p = 1, \dots, q\}$  is called the root subspace of the eigenvalue  $\lambda_0$ .

Such a CSGE always exists and, as with usual Jordan chains for matrices, condition (ii)(3) guarantees that chains of highest order are taken first, so that  $\mu_1 \geq \mu_2 \geq \dots \geq \mu_p$  holds.

Next we state the general formula of Keldysh ([MM, Thm.1.6.5]) for a finite number of eigenvalues inside a given contour (cf. [B, Cor.2.8] for this generalization).

**Theorem 2.6.** *Let  $F \in \mathbb{H}(\Omega, \mathcal{F}(\mathcal{H}, \mathcal{K}))$  and let  $\Gamma \subset \rho(F)$  be a simple closed contour with bounded interior  $\Omega_0$  in  $\Omega$ . Let  $\Omega_0 \cap \sigma(F) = \{\lambda_1, \dots, \lambda_{\varkappa}\}$  and consider for each  $\lambda_n$ ,  $n = 1, \dots, \varkappa$ , a CSGE denoted by*

$$v_{j,p}^n \in \mathcal{H}, \quad j = 0, \dots, \mu_{n,p} - 1, \quad p = 1, \dots, q_n, \quad n = 1, \dots, \varkappa.$$

*Then there exist corresponding CSGEs*

$$w_{j,p}^n \in \mathcal{K}', \quad j = 0, \dots, \mu_{n,p} - 1, \quad p = 1, \dots, q_n, \quad n = 1, \dots, \varkappa,$$

*for  $F^\top \in \mathbb{H}(\Omega, \mathcal{F}(\mathcal{K}', \mathcal{H}'))$ , an open set  $\mathcal{U}$  with  $\Omega_0 \cup \Gamma \subset \mathcal{U} \subset \Omega$  and a function  $H \in \mathbb{H}(\mathcal{U}, \mathcal{F}(\mathcal{K}, \mathcal{H}))$  such that*

$$F(\lambda)^{-1} = \sum_{n=1}^{\varkappa} \sum_{p=1}^{q_n} \sum_{j=1}^{\mu_{n,p}} (\lambda - \lambda_n)^{-j} \sum_{\nu=0}^{\mu_{n,p}-j} v_{\nu,p}^n w_{\mu_{n,p}-j-\nu}^{n\top} + H(\lambda), \quad \lambda \in \mathcal{U} \setminus \sigma(F). \quad (2.25)$$

*Remark 2.7.* The dual CSGEs are uniquely determined by an orthogonality condition which generalizes (2.7), see [MM, Thm.1.6.5]. We omit these conditions here since they will not be used in the sequel.  $\diamond$

With Keldysh' formula (2.25) and Cauchy's formula we repeat the calculations that lead to (2.13), (2.14). The result is

$$D_0 = G_l G_r^\top, \quad D_1 = G_l \Lambda G_r^\top, \quad (2.26)$$

where the matrices  $G_l, G_r$  are of size  $m \times \varkappa_0$  and  $\ell \times \varkappa_0$ , respectively with

$$\varkappa_0 = \sum_{n=1}^{\varkappa} \sum_{p=1}^{q_n} \mu_{n,p}. \quad (2.27)$$

More explicitly, we have

$$(G_l)_{j,(n,p,\nu)} = \langle \widehat{w}_j, v_{\nu,p}^n \rangle, \quad \begin{matrix} j = 1, \dots, m, \\ \nu = 0, \dots, \mu_{n,p} - 1, \end{matrix} \quad p = 1, \dots, q_n, \quad n = 1, \dots, \varkappa. \quad (2.28)$$

$$(G_r)_{j,(n,p,\nu)} = \langle w_{\mu_{n,p}-\nu-1}, \widehat{v}_j \rangle, \quad \begin{matrix} j = 1, \dots, \ell, \\ \nu = 0, \dots, \mu_{n,p} - 1, \end{matrix} \quad p = 1, \dots, q_n, \quad n = 1, \dots, \varkappa, \quad (2.29)$$

where one may think of the triples  $(n, p, \nu)$  being ordered lexicographically. Moreover, it turns out that  $\Lambda$  is of Jordan normal form

$$\Lambda = \begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_{\varkappa} \end{pmatrix}, \quad J_n = \begin{pmatrix} J_{n,1} & & \\ & \ddots & \\ & & J_{n,q_n} \end{pmatrix}, \quad (2.30)$$

$$J_{n,p} = \begin{pmatrix} \lambda_n & 1 & & \\ & \ddots & \ddots & \\ & & \lambda_n & 1 \\ & & & \lambda_n \end{pmatrix} \in \mathbb{C}^{\mu_{n,p} \times \mu_{n,p}}. \quad (2.31)$$



This immediately leads to the following generalization of Theorem 2.4.

**Theorem 2.8.** *Let  $F \in \mathbb{H}(\Omega, \mathcal{F}(\mathcal{H}, \mathcal{K}))$  and let  $\Gamma \subset \rho(F)$  be a simple closed contour with bounded interior  $\Omega_0$  in  $\Omega$ . Let  $\Omega_0 \cap \sigma(F) = \{\lambda_1, \dots, \lambda_\varkappa\}$  and denote the CSGEs associated with  $\lambda_n$  by*

$$v_{j,p}^n, \quad j = 0, \dots, \mu_{n,p} - 1, \quad p = 1, \dots, q_n, \quad n = 1, \dots, \varkappa.$$

*Consider linearly independent elements  $\hat{w}_j \in \mathcal{H}'$ ,  $j = 1, \dots, m$ , and  $\hat{v}_k \in \mathcal{H}$ ,  $k = 1, \dots, \ell$ , such that  $m, \ell \geq \varkappa_0$ , see (2.27). Determine the matrices  $D_0, D_1 \in \mathbb{C}^{m \times \ell}$  by (2.3), (2.4), (2.5) and assume that the matrices  $G_l, G_r$  from (2.28), (2.29) have maximum rank. Then the  $\varkappa_0 \times \varkappa_0$ -matrix  $D = V_0^* D_1 W_0 \Sigma_0^{-1}$ , where  $V_0, \Sigma_0, W_0$  are given by the singular value decomposition (2.16) of  $D_0$ , has Jordan normal form (2.30), (2.31) which coincides with the multiplicity structure of the spectrum of the nonlinear operator inside  $\Gamma$ .*

*Remarks 2.9.* (a) It is well known that the Jordan normal form is not robust to perturbations and, therefore, not a suitable object for numerical computations. Therefore, it seems questionable, whether one should compute approximate CSGEs by applying formula (2.23) to the columns of  $G_l$  in (2.28). Nevertheless, Theorem 2.8 has some significance. In our case perturbations of the matrices  $D_0, D_1, V_0, W_0, \Sigma_0$  may be caused by approximate solutions of the operator equations (2.3), by quadrature errors for the contour integrals (2.5), or by errors of the singular value decomposition (2.16) (see Section 2.4 for more details). This leads to a perturbed matrix  $D$  with well known spectral properties, e.g. the invariant subspaces of  $D$  belonging to the eigenvalues  $\lambda_n, n = 1, \dots, \varkappa$ , keep their dimension and perturb with the same order, see [StS], and the error of a single eigenvalue  $\lambda_n$  grows at most with the power  $1/\mu_{1,n}$  of the perturbation, where  $\mu_{1,n}$  is the maximum rank of eigenvectors belonging to  $\lambda_n$  (cf. [K]).

(b) There are several reasons that may cause a rank defect for the matrices  $G_l, G_r$  in (2.11) and (2.28), (2.29), respectively. First, it is possible that  $F$  has more eigenvalues inside the contour than the dimension of the space  $\mathcal{H}$ . For example, this is typical for characteristic equations of ordinary delay equations. However, this cannot occur with infinite dimensional spaces  $\mathcal{H}, \mathcal{K}$  which is our main concern here. Second, there may be a vector  $\sum_{n=1}^{\varkappa} \alpha_n v_n$  that is annihilated by all test functionals  $\hat{w}_j$  or a functional  $\sum_{j=1}^m \beta_j w_j$  that annihilates all test functions  $\hat{v}_k$  (similarly for CSGEs). If the eigenfunctions  $v_n$  and  $w_n$  are linearly independent and the data  $\hat{w}_j, \hat{v}_k$  are chosen at random we consider this case to be nongeneric. However, it is possible for nonlinear eigenvalue problems that eigenvectors belonging to different eigenvalues are linearly dependent (see [B, Sec.4,5] for such an example). The contour method can be extended to handle all these degenerate cases by evaluating higher order moments

$$\frac{1}{2\pi i} \int_{\Gamma} \lambda^\nu E(\lambda) d\lambda, \quad \nu = 0, \dots, 2K - 1.$$

It is shown in [B, Lem.5.1] that it suffices to take  $K = \sum_{n=1}^{\varkappa} \mu_{n,1}$  under the conditions of Theorem 2.8.  $\diamond$

**2.4. Approximation of operators.** We consider a sequence of approximate operators  $F_N \in \mathbb{H}(\Omega, \mathcal{F}(\mathcal{H}_N, \mathcal{K}_N))$ ,  $N \in \mathbb{N}$ , and study the errors for the linear system (2.3) and the spectrum  $\sigma(F)$  when  $F$  is replaced by  $F_N$ . We will use the framework of discrete approximations (see [V76] and [V80] for an English reference) which

has the advantage that  $\mathcal{H}_N, \mathcal{K}_N$  are general Banach spaces (not necessarily subspaces of  $\mathcal{H}, \mathcal{K}$ ) connected to  $\mathcal{H}, \mathcal{K}$  only via a set of linear operators (not necessarily projections)

$$p_N : \mathcal{H} \mapsto \mathcal{H}_N, \quad q_N : \mathcal{K} \mapsto \mathcal{K}_N, \quad N \in \mathbb{N}.$$

In Section 4 we will apply the theory to boundary value problems on the infinite line when approximated by two-point boundary value problems on a bounded interval. In order to assist readers unfamiliar with the theory we impose conditions slightly stronger than necessary, and we try to avoid as many notions as possible from [V76]. Our assumptions are as follows:

- (D1) There exist Banach spaces  $\mathcal{H}_N, \mathcal{K}_N, N \in \mathbb{N}$  and linear bounded mappings  $p_N \in \mathcal{L}(\mathcal{H}, \mathcal{H}_N), q_N \in \mathcal{L}(\mathcal{K}, \mathcal{K}_N)$  with the property

$$\lim_{N \rightarrow \infty} \|p_N v\|_{\mathcal{H}_N} = \|v\|_{\mathcal{H}}, \quad v \in \mathcal{H}, \quad \lim_{N \rightarrow \infty} \|q_N v\|_{\mathcal{K}_N} = \|v\|_{\mathcal{K}}, \quad v \in \mathcal{K}.$$

- (D2) Given  $F \in \mathbb{H}(\Omega, \mathcal{F}(\mathcal{H}, \mathcal{K}))$  with  $\rho(F) \neq \emptyset$  and  $F_N \in \mathbb{H}(\Omega, \mathcal{F}(\mathcal{H}_N, \mathcal{K}_N)), N \in \mathbb{N}$  with  $\sup_{N \in \mathbb{N}} \sup_{\lambda \in \mathcal{C}} \|F_N(\lambda)\| < \infty$  for every compact set  $\mathcal{C} \subset \Omega$ .

- (D3)  $F_N(\lambda)$  converges regularly to  $F(\lambda)$  for all  $\lambda \in \Omega$  in the following sense:

- (a)  $\lim_{N \rightarrow \infty} \|F_N(\lambda)p_N v - q_N F(\lambda)v\| = 0, \quad v \in \mathcal{H},$
- (b) for any subsequence  $v_N \in \mathcal{H}_N, N \in \mathbb{N}' \subset \mathbb{N}$  with  $\|v_N\|_{\mathcal{H}_N}, N \in \mathbb{N}'$  bounded and  $\lim_{N \in \mathbb{N}' \rightarrow \infty} \|F_N(\lambda)v_N - q_N y\|_{\mathcal{K}_N} = 0$  for some  $y \in \mathcal{K}$ , there exists a subsequence  $\mathbb{N}'' \subset \mathbb{N}'$  and a  $v \in \mathcal{H}$  such that  $\lim_{N \in \mathbb{N}'' \rightarrow \infty} \|v_N - p_N v\|_{\mathcal{H}_N} = 0.$

The term  $\|F_N(\lambda)p_N v - q_N F(\lambda)v\|$  is called the *consistency error* since it measures the defect in the noncommuting diagram.

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{F(\lambda)} & \mathcal{K} \\ p_N \downarrow & & \downarrow q_N \\ \mathcal{H}_N & \xrightarrow{F_N(\lambda)} & \mathcal{K}_N \end{array} \quad (2.32)$$

The following theorem summarizes convergence results from [V76, §3(3), §4(33)].

**Theorem 2.10.** *Under the assumptions (D1)-(D3) the following assertions hold:*

- (i) *For any compact set  $\mathcal{C} \subset \rho(F)$  and any  $\hat{v} \in \mathcal{K}$  there exists an  $N_0 \in \mathbb{N}$  such that the linear equation  $F_N(\lambda)y_N = q_N \hat{v}$  has a unique solution  $y_N = y_N(\lambda)$  for  $N \geq N_0, \lambda \in \mathcal{C}$ , and the following estimate holds*

$$\sup_{\lambda \in \mathcal{C}} \|y_N(\lambda) - p_N y(\lambda)\|_{\mathcal{H}_N} \leq C \sup_{\lambda \in \mathcal{C}} \|F_N(\lambda)p_N y(\lambda) - q_N F(\lambda)y(\lambda)\|, \quad (2.33)$$

where  $y(\lambda) \in \mathcal{H}$  is the unique solution to  $F(\lambda)y = \hat{v}$ .

- (ii) *For any  $\lambda_0 \in \sigma(F)$  there exists  $N_0 \in \mathbb{N}$  and a sequence  $\lambda_N \in \sigma(F_N), N \geq N_0$ , such that  $\lambda_N \rightarrow \lambda_0$  as  $N \rightarrow \infty$ . For any sequence  $\lambda_N \in \sigma(F_N)$  with this convergence property, and associated eigenvectors  $v_N^0 \in \mathcal{N}(F_N(\lambda_N)), \|v_N^0\|_{\mathcal{H}_N} = 1$ , the following estimates hold:*

$$|\lambda_N - \lambda_0| \leq C \varepsilon_N^{\frac{1}{r_0}} \quad (2.34)$$

$$\inf_{v_0 \in \mathcal{N}(F(\lambda_0))} \|v_N^0 - p_N v_0\|_{\mathcal{H}_N} \leq C \varepsilon_N^{\frac{1}{r_0}}, \quad (2.35)$$

where  $r_0 = \mu_1$  is the maximum rank of eigenvectors that belong to  $\lambda_0$ . The quantity  $\varepsilon_N$  is a consistency error defined by

$$\varepsilon_N = \max_{|\lambda - \lambda_0| \leq \delta} \max_{v \in \mathcal{M}} \|F_N(\lambda)p_N v - q_N F(\lambda)v\|,$$

where  $\mathcal{M} = \text{span}\{v_{j,p} : j = 0, \dots, \mu_p - 1, p = 1, \dots, q\}$  is the root space associated with  $\lambda_0$  (see Definition 2.5) and  $\delta > 0$  is chosen sufficiently small.

*Remark 2.11.* We note that our assumptions (D2) and (D3) imply regular convergence in the sense of [V76, §2(17)]. Moreover, it is easy to see that the convergence result in [V76, §3(3)] holds uniformly in  $\lambda \in \mathcal{C}$ . Of course, for the latter result continuity of  $\lambda \mapsto F(\lambda)$  is sufficient.  $\diamond$

### 3. RELATION TO THE EVANS FUNCTION

In this section we relate the abstract approach from section 2 to the study of zeroes of the Evans function. Recall from the abstract setting formula (2.12), which we rewrite as

$$E(\lambda) = \frac{1}{\mathcal{E}_z(\lambda)} H_1(\lambda) + H_0(\lambda), \quad \lambda \in \Omega \setminus \{\lambda_1, \dots, \lambda_\varkappa\}, \quad (3.1)$$

where we have introduced

$$\mathcal{E}_z(\lambda) = \prod_{j=1}^{\varkappa} (\lambda - \lambda_j), \quad H_1(\lambda) = G_l \text{diag} \left( \prod_{\nu=1, \nu \neq j}^{\varkappa} (\lambda - \lambda_\nu), j = 1, \dots, \varkappa \right) G_r^\top.$$

Note that  $\mathcal{E}_z(\lambda)$  may be viewed as an abstract version of the Evans function since its zeroes are exactly the eigenvalues of  $F(\lambda)$ . Moreover,  $H_0, H_1$  are holomorphic matrix functions such that  $H_1$  degenerates to a rank one matrix at every simple eigenvalue.

In the following we make the above relation more explicit when the classical definition of the Evans function is used, that is as the determinant of a matrix with holomorphic columns determined from appropriate subspaces. In the first step we set up a normalized Evans function that is independent of a perturbative situation.

**3.1. Definition of a normalized Evans function.** In general the Evans function may be considered as a determinant that measures the coalescence of two families of subspaces depending holomorphically on a parameter. We define a normalized version of the function that is unique up to a sign. To begin, we introduce the class of matrices

$$\mathbb{M}^{d,k} = \{P \in \mathbb{C}^{d,k} : \det(P^\top P) = 1\} \quad (3.2)$$

and recall the following elementary fact: Let  $U \subset \mathbb{C}^d$  be a subspace of dimension  $k$ . Then two matrices  $P_1, P_2 \in \mathbb{C}^{d,k}$  of rank  $k$  satisfy  $\mathcal{R}(P_1) = \mathcal{R}(P_2) = U$  if and only if there is an invertible matrix  $R \in \mathbb{C}^{k,k}$  such that  $P_2 = P_1 R$ .

**Definition 3.1.** We call two matrices  $P_1, P_2 \in \mathbb{C}^{d,k}$  equivalent and write  $P_1 \approx P_2$  provided  $P_2 = P_1 R$  for some  $R \in \mathbb{C}^{k,k}$  such that  $\det(R) = -1$  or  $\det(R) = 1$ . We will use notation  $[P] = \{P_1 \in \mathbb{C}^{d,k} : P_1 \approx P\}$  for the equivalence class.

*Remark 3.2.* If  $P_1, P_2 \in \mathbb{M}^{d,k}$  and  $\mathcal{R}(P_1) = \mathcal{R}(P_2)$  then  $P_1 \approx P_2$  holds automatically since  $1 = \det(P_2^\top P_2) = \det(P_1^\top P_1) \det(R)^2 = \det(R)^2$ .  $\diamond$

**Lemma 3.3.** *For any rank  $k$  projection  $\Pi$  in  $\mathbb{C}^d$  there exist  $P, \Phi \in \mathbb{C}^{d,k}$  such that*

$$\Pi = P\Phi^\top, \quad \Phi^\top P = I_k. \quad (3.3)$$

*Moreover, if the inequality  $\det(P^\top P) \neq 0$  holds for some  $P$  from (3.3), then it holds for any such  $P$ , and there exist  $P_0, \Phi_0 \in \mathbb{C}^{d,k}$  such that*

$$\Pi = P_0\Phi_0^\top, \quad P_0 \in \mathbb{M}^{d,k}, \quad \Phi_0^\top P_0 = I_k. \quad (3.4)$$

*Proof.* Using basis vectors in  $\mathcal{R}(\Pi)$  and  $\mathcal{R}(\Pi^\top)$  as columns, we obtain rank  $k$  matrices  $P, \tilde{\Phi} \in \mathbb{C}^{d,k}$  with  $\mathcal{R}(\Pi) = \mathcal{R}(P)$  and  $\mathcal{R}(\Pi^\top) = \mathcal{R}(\tilde{\Phi})$ , respectively. Standard linear algebra shows  $\mathcal{N}(\tilde{\Phi}^\top) = \mathcal{R}(\tilde{\Phi})^\perp = \mathcal{R}(\Pi^\top)^\perp = \mathcal{N}(\Pi)$ , where  $^\perp$  means orthogonal with respect to the duality pairing (see (2.1) in Section 2).

First we obtain invertibility of  $\tilde{\Phi}^\top P \in \mathbb{C}^{k,k}$ : For  $c \in \mathbb{C}^k$  the assumption  $\tilde{\Phi}^\top Pc = 0$  implies  $Pc \in \mathcal{R}(P) \cap \mathcal{N}(\tilde{\Phi}^\top)^\perp = \mathcal{R}(\Pi) \cap \mathcal{N}(\Pi) = \{0\}$  and hence  $c = 0$  since  $P$  has full rank. Second, we claim that

$$\Pi = P(\tilde{\Phi}^\top P)^{-1}\tilde{\Phi}^\top, \quad P, \tilde{\Phi} \in \mathbb{C}^{d,k}. \quad (3.5)$$

Indeed, the matrix  $\tilde{\Pi} = P(\tilde{\Phi}^\top P)^{-1}\tilde{\Phi}^\top$  is a projection with  $\mathcal{R}(\tilde{\Pi}) = \mathcal{R}(P) = \mathcal{R}(\Pi)$  and  $\mathcal{N}(\tilde{\Pi}) = \mathcal{N}(\tilde{\Phi}^\top) = \mathcal{N}(\Pi)$ , yielding  $\Pi = \tilde{\Pi}$  and finishing the proof of (3.5). Letting

$$\Phi = \tilde{\Phi}(P^\top \tilde{\Phi})^{-1}, \quad (3.6)$$

we arrive at the representation (3.3).

If  $P, \tilde{P}$  are two matrices as in (3.3), then  $P = \tilde{P}R$  for some invertible  $R \in \mathbb{C}^{k,k}$  due to  $\mathcal{R}(P) = \mathcal{R}(\tilde{P}) = \mathcal{R}(\Pi)$ . Now  $\det(P^\top P) = \det(\tilde{P}^\top \tilde{P}) \det(R)^2$  proves the second assertion in the lemma. Finally, if  $\det(P^\top P) \neq 0$  then we can replace  $P$  and  $\tilde{\Phi}$  in (3.5) by

$$P_0 = P \operatorname{diag}(\det(P^\top P)^{-1/2}, 1, \dots, 1) \in \mathbb{M}^{d,k} \quad \text{and} \quad \Phi_0 = \tilde{\Phi}(P_0^\top \tilde{\Phi})^{-1}. \quad (3.7)$$

Since  $P_0 \in \mathbb{M}^{d,k}$ , we arrive at (3.4).  $\blacksquare$

Next consider two subspaces of complementary dimension. We may write them as images of projections that both have the representation (3.3), or, perhaps, even the normalized representation (3.4).

**Definition 3.4.** *Let  $U, V \subset \mathbb{C}^d$  be subspaces of dimension  $k$  and  $d-k$ , respectively. Pick any  $P \in \mathbb{C}^{d,k}$  of rank  $k$  such that  $\mathcal{R}(P) = U$  and  $Q \in \mathbb{C}^{d,d-k}$  of rank  $d-k$  such that  $\mathcal{R}(Q) = V$ . Then we call*

$$\mathcal{D}([P], [Q]) = \{\det(P_1|Q_1) : P_1 \in [P], Q_1 \in [Q]\} \quad (3.8)$$

*the determinant set of the equivalent classes  $([P], [Q])$  from Definition 3.1. In addition, assume that  $P \in \mathbb{M}^{d,k}, Q \in \mathbb{M}^{d,d-k}$ . Then we call*

$$\mathcal{D}(U, V) = \{\det(P|Q) : P \in \mathbb{M}^{d,k}, Q \in \mathbb{M}^{d,d-k}, \mathcal{R}(P) = U, \mathcal{R}(Q) = V\} \quad (3.9)$$

*the determinant set of the subspaces  $U$  and  $V$ .*

**Remark 3.5.** We stress that the set  $\mathcal{D}([P], [Q])$  is defined with no additional assumptions on the subspaces  $U, V$ . Although this set does not depend on the choice of the representatives  $P, Q$  in the equivalence classes  $[P], [Q]$ , it does depend on the choice of the equivalence classes. On the other hand, the set  $\mathcal{D}(U, V)$  is defined under an additional assumption on the subspaces  $U, V$ , but is uniquely determined by these subspaces.  $\diamond$

Our key observation is the following lemma which shows that the class (3.2) and the definitions (3.8), (3.9) have some significance.

**Lemma 3.6.** *Let  $U, V \subset \mathbb{C}^d$  be subspaces of dimension  $k$  and  $d - k$ , respectively. Pick any  $P \in \mathbb{C}^{d,k}$  of rank  $k$  such that  $\mathcal{R}(P) = U$  and  $Q \in \mathbb{C}^{d,d-k}$  of rank  $d - k$  such that  $\mathcal{R}(Q) = V$ . Then there exists  $z \in \mathbb{C}$  such that  $\mathcal{D}([P], [Q]) = \{z, -z\}$ . The subspaces  $U$  and  $V$  are complementary if and only if  $z \neq 0$ . In addition, assume that there are  $P_0, Q_0$  picked as above and such that  $\det(P_0^\top P_0) \det(Q_0^\top Q_0) \neq 0$ . Then there exists  $z \in \mathbb{C}$  such that  $\mathcal{D}(U, V) = \{z, -z\}$ .*

*Proof.* First take  $P_1 \in [P]$ ,  $Q_1 \in [Q]$  with  $\mathcal{R}(P_1) = U$ ,  $\mathcal{R}(Q_1) = V$  and define  $z = \det(P_1|Q_1)$ . Then take another pair  $P_2, Q_2$  of this type and use Definition 3.1 to write  $P_2 = P_1 R$ ,  $Q_2 = Q_1 S$  for some matrices  $R \in \mathbb{C}^{k,k}$ ,  $S \in \mathbb{C}^{d-k,d-k}$  such that  $\det(R), \det(S) \in \{-1, 1\}$ . Therefore we conclude

$$\det(P_2|Q_2) = \det \left( (P_1|Q_1) \begin{pmatrix} R & 0 \\ 0 & S \end{pmatrix} \right) = z \det(R) \det(S).$$

This proves  $\mathcal{D}([P], [Q]) \subset \{z, -z\}$ . In fact we have equality since  $-z$  is attained by taking  $R = \text{diag}(-1, 1, \dots, 1)$ ,  $S = I_{d-k}$ . The second assertion of the lemma is obvious. By normalizing  $P_0, Q_0$  as in (3.7), we may assume that  $P_0 \in \mathbb{M}^{d,k}$ ,  $Q_0 \in \mathbb{M}^{d,d-k}$ , and the third assertion holds by Remark 3.2.  $\blacksquare$

**Definition 3.7.** *A family of subspaces  $U(\lambda) \subset \mathbb{C}^d$ ,  $\lambda \in \Omega \subset \mathbb{C}$ , is called holomorphic in  $\Omega$  if there exists a holomorphic family of projections  $\Pi \in \mathbb{H}(\Omega, \mathbb{C}^{d,d})$  such that  $\mathcal{R}(\Pi(\lambda)) = U(\lambda)$  for all  $\lambda \in \Omega$ .*

Clearly, by the connectedness of  $\Omega$  the projections must be of constant rank and hence the subspaces are of constant dimension.

The following lemma generalizes the representation (3.3) and the normalized representation (3.4) to holomorphic families. It is essential that the domain is simply connected.

**Lemma 3.8.** *Let  $\Omega$  be a simply connected domain in  $\mathbb{C}$ . For any holomorphic family of projections  $\Pi \in \mathbb{H}(\Omega, \mathbb{C}^{d,d})$  of rank  $k$  there exist functions  $P, \Phi \in \mathbb{H}(\Omega, \mathbb{C}^{d,k})$  such that*

$$\Pi(\lambda) = P(\lambda)\Phi(\lambda)^\top, \quad \Phi(\lambda)^\top P(\lambda) = I_k, \quad \text{for all } \lambda \in \Omega. \quad (3.10)$$

*Moreover, for any  $P \in \mathbb{H}(\Omega, \mathbb{C}^{d,k})$  such that  $\mathcal{R}(P(\lambda)) = \mathcal{R}(\Pi(\lambda))$  for all  $\lambda \in \Omega$ , the set*

$$\Lambda = \{\lambda \in \Omega : \det(P(\lambda)^\top P(\lambda)) = 0\} \quad (3.11)$$

*is discrete, may be empty, and is independent of the choice of  $P$ . Finally, there exist functions  $P_0, \Phi_0 \in \mathbb{H}(\Omega \setminus \Lambda, \mathbb{C}^{d,k})$  such that*

$$\Pi(\lambda) = P_0(\lambda)\Phi_0(\lambda)^\top, \quad \Phi_0(\lambda)^\top P_0(\lambda) = I_k, \quad P_0(\lambda) \in \mathbb{M}^{d,k} \text{ for all } \lambda \in \Omega \setminus \Lambda. \quad (3.12)$$

*Proof.* By a result from [K, Sec.II.4.2] there exist functions  $P, \tilde{\Phi} \in \mathbb{H}(\Omega, \mathbb{C}^{d,k})$  such that  $\mathcal{R}(\Pi(\lambda)) = \mathcal{R}(P(\lambda))$ ,  $\mathcal{R}(\Pi(\lambda)^\top) = \mathcal{R}(\tilde{\Phi}(\lambda))$  for all  $\lambda \in \Omega$ . This step uses simple connectedness. In the next step, as in the proof of Lemma 3.3, we normalize  $\tilde{\Phi}(\lambda)$  as in (3.6), and keep holomorphy. This proves (3.10). The set of zeros of the holomorphic function  $\det(P(\cdot)^\top P(\cdot))$  is discrete. If  $P_1, P_2 \in \mathbb{H}(\Omega, \mathbb{C}^{d,k})$  are such that  $\mathcal{R}(P_1(\lambda)) = \mathcal{R}(P_2(\lambda)) = \mathcal{R}(\Pi(\lambda))$  for all  $\lambda \in \Omega$  then there is a non-singular  $R(\lambda) \in \mathbb{C}^{k,k}$  such that  $P_2(\lambda) = P_1(\lambda)R(\lambda)$ ; hence,  $\det(P_1(\cdot)^\top P_1(\cdot))$  and  $\det(P_2(\cdot)^\top P_2(\cdot))$  have the same zeros. If  $\lambda \in \Omega \setminus \Lambda$  then  $\det(P(\lambda)^\top P(\lambda)) \neq 0$  and we

normalize  $P(\lambda)$  and  $\Phi(\lambda)$  as in (3.7), and keep holomorphy. Note that the square root has (up to a sign) a unique analytic continuation in any simply connected domain that does not contain 0.  $\blacksquare$

The following theorem shows how one can assign to two families of holomorphic subspaces the *usual* Evans function which is equal to zero if and only if the subspaces are not complementary. This function is holomorphic on all of  $\Omega$  see (3.13). It depends, up to a sign, on the equivalence classes from Definition 3.1 for the matrices formed by the basis vectors of the subspaces. In addition, one can assign to the two families of subspaces an Evans function that is unique, up to a sign, but is holomorphic on a smaller set  $\Omega \setminus \Lambda(U, V)$ , see (3.15). We will call it the *normalized* Evans function. We refer to Remark 3.5 regarding the sets  $\mathcal{D}([P(\lambda)], [Q(\lambda)])$  and  $\mathcal{D}(U(\lambda), V(\lambda))$  used in (3.13), (3.15) below.

**Theorem 3.9.** *Let  $\Omega \in \mathbb{C}$  be a simply connected domain and let  $U(\lambda), V(\lambda), \lambda \in \Omega$ , be two holomorphic families of subspaces of dimensions  $k$  and  $d - k$ , respectively. Pick any  $P \in \mathbb{H}(\Omega, \mathbb{C}^{d,k})$  such that  $\mathcal{R}(P(\lambda)) = U(\lambda)$ ,  $\text{rank}(P(\lambda)) = k$ , and  $Q \in \mathbb{H}(\Omega, \mathbb{C}^{d,d-k})$  such that  $\mathcal{R}(Q(\lambda)) = V(\lambda)$ ,  $\text{rank}(Q(\lambda)) = d - k$  for all  $\lambda \in \Omega$ . Then the following assertions hold.*

(i) *There exists a holomorphic function  $\mathcal{E} : \Omega \rightarrow \mathbb{C}$  such that*

$$\mathcal{E}(\lambda) \in \mathcal{D}([P(\lambda)], [Q(\lambda)]) \quad \text{for all } \lambda \in \Omega. \quad (3.13)$$

(ii)  *$\mathcal{E}(\lambda_0) = 0$  if and only if  $U(\lambda_0) \cap V(\lambda_0) \neq \{0\}$ .*

(iii) *If  $\mathcal{E}_1 \in \mathbb{H}(\Omega, \mathbb{C})$  is any function satisfying (3.13) then either  $\mathcal{E}_1 = \mathcal{E}$  or  $\mathcal{E}_1 = -\mathcal{E}$ .*

(iv) *Moreover, the set*

$$\Lambda(U, V) = \{\lambda \in \Omega : \det(P(\lambda)^\top P(\lambda)) \det(Q(\lambda)^\top Q(\lambda)) = 0\} \quad (3.14)$$

*is discrete, may be empty, and is independent of the choice of  $P, Q$  picked as above.*

(v) *Finally, there exists a holomorphic function  $\mathcal{E}_0 : \Omega \setminus \Lambda(U, V) \rightarrow \mathbb{C}$  such that*

$$\mathcal{E}_0(\lambda) \in \mathcal{D}(U(\lambda), V(\lambda)) \quad \text{for all } \lambda \in \Omega \setminus \Lambda(U, V), \quad (3.15)$$

*and properties (i), (ii) hold for  $\mathcal{E}_0$ .*

*Proof.* By Lemma 3.8 we have a representation for the projections  $\Pi_U, \Pi_V \in \mathbb{H}(\Omega, \mathbb{C}^{d,d})$  associated with  $U(\lambda), V(\lambda)$  as follows

$$\Pi_U(\lambda) = P(\lambda)\Phi(\lambda)^\top, \quad \Pi_V(\lambda) = Q(\lambda)\Psi(\lambda)^\top, \quad \lambda \in \Omega, \quad (3.16)$$

where  $P, \Phi \in \mathbb{H}(\Omega, \mathbb{C}^{d,k})$ ,  $Q, \Psi \in \mathbb{H}(\Omega, \mathbb{C}^{d,d-k})$  and for all  $\lambda \in \Omega$

$$\Phi^\top(\lambda)P(\lambda) = I_k, \quad \Psi^\top(\lambda)Q(\lambda) = I_{d-k}. \quad (3.17)$$

Defining

$$\mathcal{E}(\lambda) = \det(P(\lambda)|Q(\lambda)), \quad \lambda \in \Omega, \quad (3.18)$$

and using Lemma 3.6 proves assertions (i) and (ii). Now let  $\mathcal{E}_1$  be any function with the properties of  $\mathcal{E}$ . Then the quotient  $\mathcal{E}_1/\mathcal{E}$  is holomorphic on  $\Omega \setminus \mathcal{N}(U, V)$  where the zero set is given by

$$\mathcal{N}(U, V) = \{\lambda \in \Omega : \mathcal{D}([P(\lambda)], [Q(\lambda)]) = \{0\}\} = \{\lambda \in \Omega : \mathcal{E}(\lambda) = 0\}. \quad (3.19)$$

Since the quotient assumes only values  $\pm 1$  there and  $\mathcal{N}(U, V)$  contains only isolated points it is a constant yielding (iii). Assertion (iv) follows from Lemma 3.8, see (3.10). Also, by Lemma 3.8, see (3.12), we can choose functions  $P_0, \Phi_0 \in \mathbb{H}(\Omega \setminus$

$\Lambda(U, V), \mathbb{C}^{d,k}$ ) and  $Q_0, \Psi_0 \in \mathbb{H}(\Omega \setminus \Lambda(U, V), \mathbb{C}^{d,d-k})$  such that, in addition to (3.16), (3.17), the following normalization holds:

$$P_0(\lambda) \in \mathbb{M}^{d,k}, \quad Q_0(\lambda) \in \mathbb{M}^{d,d-k}, \quad \lambda \in \Omega \setminus \Lambda(U, V). \quad (3.20)$$

Defining

$$\mathcal{E}_0(\lambda) = \det(P_0(\lambda)|Q_0(\lambda)), \quad \lambda \in \Omega \setminus \Lambda(U, V), \quad (3.21)$$

and using Lemma 3.6 proves assertion (v).  $\blacksquare$

*Remark 3.10.* 1. An alternative proof of the theorem can be obtained directly from Lemma 3.6 without using Lemma 3.8 and [K, Sec.II.4.2]. One first defines  $\mathcal{E}$ , respectively,  $\mathcal{E}_0$  locally via (3.18), respectively, (3.21) using a respective local normalization that is always possible (without simple connectedness). Then one shows that one can continue holomorphically along any curve in  $\Omega$ , respectively,  $\Omega \setminus \Lambda(U, V)$  (since the set  $\mathcal{N}(U, V)$  defined in (3.19) is isolated) and the sign of the continuation is determined from Lemma 3.6. Since the domain is simply connected, the result follows from the Monodromy Theorem, see, e.g., [C, Cor.IX.3.9].

2. According to Theorem 3.9 the normalized Evans function is unique up to a sign on simply connected domains. The assumption of simple connectedness is crucial for the proof. The main ingredient which makes the Evans function unique up to a sign was to allow only matrices with  $\det(P^\top P) = 1$  in Definition 3.4. When homotopy arguments are applied to the Evans functions between different points in  $\Omega$ , e.g. between 0 and  $\infty$ , it remains to be checked whether the values are from the same leaf that belongs to one of the signs.

3. Normalization (3.7) shows that the normalized Evans function  $\mathcal{E}_0$  can not be continued to  $\Lambda(U, V)$  since  $\mathcal{E}_0(\lambda) \sim (\lambda - \lambda_0)^{-m/2}$  as  $\lambda \rightarrow \lambda_0$  at any point  $\lambda_0 \in \Lambda(U, V)$  which is a zero of order  $m$  of the function  $\det(P(\cdot)^\top P(\cdot)) \det(Q(\cdot)^\top Q(\cdot))$  holomorphic in  $\Omega$ .  $\diamond$

In the next step we project vectors in  $\mathbb{C}^d$  onto their components in the subspaces  $U(\lambda)$  and  $V(\lambda)$ . Although this cannot work at the zeroes of the Evans function we insist that the main part of the projection stays holomorphic at the singularity. Recall the adjugate  $\text{adj}(A)$  of a square matrix  $A \in \mathbb{C}^{d,d}$  given by

$$(\text{adj}(A))_{ij} = (-1)^{i+j} \det(A_{\ell=1, \dots, j-1, j+1, \dots, d}^{m=1, \dots, i-1, i+1, \dots, d}), \quad (3.22)$$

(see e.g. [S, Sec.4.4]). It satisfies the identities  $A \text{adj}(A) = \text{adj}(A)A = \det(A)I_d$  and  $\det(\text{adj}(A)) = \det(A)^{d-1}$ . More importantly, by definition the adjugate preserves holomorphy, i.e. if  $A$  is in  $\mathbb{H}(\Omega, \mathbb{C}^{d,d})$  then so is  $\text{adj}(A)$ .

**Theorem 3.11.** *Let the assumptions of Theorem 3.9 hold and let  $\mathcal{E} \in \mathbb{H}(\Omega, \mathbb{C})$ , respectively,  $\mathcal{E}_0 \in \mathbb{H}(\Omega \setminus \Lambda(U, V), \mathbb{C})$  be one of the two Evans functions, respectively, normalized Evans functions, determined there. Then there exist matrix valued functions  $\mathcal{Y}_U, \mathcal{Y}_V \in \mathbb{H}(\Omega, \mathbb{C}^{d,d})$  with the following properties:*

$$\mathcal{E}(\lambda)I_d = \mathcal{Y}_U(\lambda) + \mathcal{Y}_V(\lambda) \quad \text{for all } \lambda \in \Omega, \quad (3.23)$$

$$\begin{aligned} \mathcal{R}(\mathcal{Y}_U(\lambda)) &= U(\lambda), \quad \mathcal{N}(\mathcal{Y}_U(\lambda)) = V(\lambda), \\ \mathcal{R}(\mathcal{Y}_V(\lambda)) &= V(\lambda), \quad \mathcal{N}(\mathcal{Y}_V(\lambda)) = U(\lambda) \end{aligned} \quad \text{if } \mathcal{E}(\lambda) \neq 0, \quad (3.24)$$

$$\begin{aligned} \mathcal{R}(\mathcal{Y}_U(\lambda)) &\subset U(\lambda), \quad \mathcal{N}(\mathcal{Y}_U(\lambda)) \supset V(\lambda), \\ \mathcal{R}(\mathcal{Y}_V(\lambda)) &\subset V(\lambda), \quad \mathcal{N}(\mathcal{Y}_V(\lambda)) \supset U(\lambda) \end{aligned} \quad \text{if } \mathcal{E}(\lambda) = 0. \quad (3.25)$$

Conversely, the functions  $\mathcal{Y}_U, \mathcal{Y}_V \in \mathbb{H}(\Omega, \mathbb{C}^{d,d})$  are uniquely determined by properties (3.23) and (3.24). Similarly, for the normalized Evans function  $\mathcal{E}_0$  there exist  $\mathcal{Y}_U^0(\lambda), \mathcal{Y}_V^0(\lambda) \in \mathbb{H}(\Omega \setminus \Lambda(U, V), \mathbb{C}^{d,d})$  with the same properties.

*Remark 3.12.* Note that  $\mathcal{Y}_U$  and  $\mathcal{Y}_V$  behave almost like projections since (3.23), (3.25) imply

$$\mathcal{Y}_U(\lambda)\mathcal{Y}_U(\lambda) = \mathcal{E}(\lambda)\mathcal{Y}_U(\lambda), \quad \mathcal{Y}_V(\lambda)\mathcal{Y}_V(\lambda) = \mathcal{E}(\lambda)\mathcal{Y}_V(\lambda), \quad \lambda \in \Omega. \quad (3.26)$$

◇

*Proof.* For the projections  $\Pi_U, \Pi_V \in \mathbb{H}(\Omega, \mathbb{C}^{d,d})$  associated with the subspaces  $U(\lambda), V(\lambda)$ , we take the representation (3.10) from Lemma 3.8 and partition the adjugate as follows

$$\text{adj}(P(\lambda)|Q(\lambda)) = \begin{pmatrix} R(\lambda)^\top \\ S(\lambda)^\top \end{pmatrix}, \quad \lambda \in \Omega, \quad (3.27)$$

where  $R \in \mathbb{H}(\Omega, \mathbb{C}^{d,k})$ ,  $S \in \mathbb{H}(\Omega, \mathbb{C}^{d,d-k})$ . With these settings we define

$$\mathcal{Y}_U(\lambda) = P(\lambda)R(\lambda)^\top, \quad \mathcal{Y}_V(\lambda) = Q(\lambda)S(\lambda)^\top \quad \text{for } \lambda \in \Omega. \quad (3.28)$$

Conditions (3.23)-(3.25) hold for  $\lambda \in \Omega$  as can be seen from the identities

$$\begin{pmatrix} P(\lambda) & Q(\lambda) \end{pmatrix} \begin{pmatrix} R(\lambda)^\top \\ S(\lambda)^\top \end{pmatrix} = \mathcal{E}(\lambda)I_d = \begin{pmatrix} R(\lambda)^\top \\ S(\lambda)^\top \end{pmatrix} \begin{pmatrix} P(\lambda) & Q(\lambda) \end{pmatrix}. \quad (3.29)$$

Note that the first equality implies (3.23) while the second implies (3.25). In case  $\mathcal{E}(\lambda) \neq 0$  the matrices  $R(\lambda), S(\lambda)$  are of full rank so that (3.24) follows.

Now consider functions  $\tilde{\mathcal{Y}}_U, \tilde{\mathcal{Y}}_V \in \mathbb{H}(\Omega, \mathbb{C}^{d,d})$  that satisfy (3.23), (3.24) in  $\Omega$ . If  $\mathcal{E}(\lambda) \neq 0$  then from the general form (3.5) and (3.24) we find representations  $\tilde{\mathcal{Y}}_U(\lambda) = P(\lambda)\tilde{R}(\lambda)^\top$ ,  $\tilde{\mathcal{Y}}_V(\lambda) = Q(\lambda)\tilde{S}(\lambda)^\top$  for some  $\tilde{R}(\lambda) \in \mathbb{C}^{d,k}$ ,  $\tilde{S}(\lambda) \in \mathbb{C}^{d,d-k}$ . Invoking (3.23) leads to

$$\begin{pmatrix} P(\lambda) & Q(\lambda) \end{pmatrix} \begin{pmatrix} \tilde{R}(\lambda)^\top \\ \tilde{S}(\lambda)^\top \end{pmatrix} = \mathcal{E}(\lambda)I_d,$$

and hence  $\tilde{R}(\lambda), \tilde{S}(\lambda)$  and  $R(\lambda), S(\lambda)$  must agree due to (3.29). Thus  $\tilde{\mathcal{Y}}_U = \mathcal{Y}_U$ ,  $\tilde{\mathcal{Y}}_V = \mathcal{Y}_V$  holds in  $\Omega \setminus \mathcal{N}(U, V)$  and hence in all of  $\Omega$  due to holomorphy.

The last assertion in the theorem follows as above by taking the representation (3.12) instead of (3.10). ■

Next we consider the behavior of the matrices  $\mathcal{Y}_U$  and  $\mathcal{Y}_V$  near zeroes of the Evans function  $\mathcal{E}$ . Let us first note, that the multiplicity of a value  $\lambda_0$  as a zero of the Evans function corresponds exactly to the algebraic multiplicity of the eigenvalue  $\lambda_0$  of the matrix pencil  $\mathcal{Y}(\lambda) = (P(\lambda)|Q(\lambda))$  from (3.18) defined in terms of root functions (see [MM, Prop.1.8.5]). For simplicity we consider only the behavior near simple eigenvalues and show that the matrices  $\mathcal{Y}_U, \mathcal{Y}_V$  degenerate to rank one matrices.

**Theorem 3.13.** *Let the assumptions of Theorem 3.11 hold and let  $\lambda_0 \in \Omega$  be a simple zero of the Evans function  $\mathcal{E}(\lambda)$  defined as in (3.18). Then there are nontrivial vectors  $v_0, w_0$  such that*

$$\mathcal{N}(P(\lambda_0)|Q(\lambda_0)) = \text{span}\{v_0\}, \quad \mathcal{N}((P(\lambda_0)|Q(\lambda_0))^\top) = \text{span}\{w_0\}.$$



Moreover, with  $v_0 = \begin{pmatrix} v_{0,1} \\ v_{0,2} \end{pmatrix}$  we have the formulas

$$\mathcal{Y}_U(\lambda_0) = \mathcal{E}'(\lambda_0)P(\lambda_0)v_{0,1}w_0^\top = -\mathcal{E}'(\lambda_0)Q(\lambda_0)v_{0,2}w_0^\top = -\mathcal{Y}_V(\lambda_0) \quad (3.30)$$

and, if  $v_0^\top v_0 = 1$ ,

$$\mathcal{E}'(\lambda_0) = v_0^\top \operatorname{adj}(P(\lambda_0)|Q(\lambda_0))(P'(\lambda_0)|Q'(\lambda_0))v_0. \quad (3.31)$$

Finally, if  $\lambda \in \Omega \setminus \Lambda(U, V)$  then the Evans function  $\mathcal{E}$  in assertions above can be replaced by the normalized Evans function  $\mathcal{E}_0$ .

*Proof.* From [MM, Prop.1.8.5] we have that the kernel of  $(P(\lambda_0)|Q(\lambda_0))$  and of its transpose are one-dimensional. Applying Keldysh's Theorem to the matrix pencil  $\mathcal{Y}(\lambda) = (P(\lambda)|Q(\lambda))$  shows that for some  $h_1 \in \mathbb{H}(\Omega_1, \mathbb{C}^{d,d})$ ,  $\Omega_1$  is some neighborhood of  $\lambda_0$ ,

$$\mathcal{Y}(\lambda)^{-1} = \frac{1}{\lambda - \lambda_0} v_0 w_0^\top + h_1(\lambda), \quad \lambda \in \Omega_1. \quad (3.32)$$

Here we have normalized  $w_0$  such that

$$1 = w_0^\top \mathcal{Y}'(\lambda_0)v_0 = w_0^\top P'(\lambda_0)v_{0,1} + w_0^\top Q'(\lambda_0)v_{0,2}.$$

Comparing the singular parts of (3.32) and

$$\mathcal{Y}(\lambda)^{-1} = \frac{1}{\mathcal{E}(\lambda)} \begin{pmatrix} R(\lambda)^\top \\ S(\lambda)^\top \end{pmatrix}$$

leads to

$$\frac{1}{\mathcal{E}'_0(\lambda_0)} \begin{pmatrix} R(\lambda_0)^\top \\ S(\lambda_0)^\top \end{pmatrix} = v_0 w_0^\top = \begin{pmatrix} v_{0,1} w_0^\top \\ v_{0,2} w_0^\top \end{pmatrix}. \quad (3.33)$$

With this and  $0 = \mathcal{Y}(\lambda_0)v_0 = P(\lambda_0)v_{0,1} + Q(\lambda_0)v_{0,2}$  we arrive at

$$\begin{aligned} \mathcal{Y}_U(\lambda_0) &= P(\lambda_0)R(\lambda_0)^\top = \mathcal{E}'_0(\lambda_0)P(\lambda_0)v_{0,1}w_0^\top \\ &= -\mathcal{E}'_0(\lambda_0)Q(\lambda_0)v_{0,2}w_0^\top = -\mathcal{Y}_V(\lambda_0). \end{aligned}$$

Finally, normalizing  $v_0^\top v_0 = 1$  we obtain (3.31) from (3.33) and the normalizing condition. Note that (3.31) may also be derived directly by differentiating  $\operatorname{adj}(\mathcal{Y}(\lambda))\mathcal{Y}(\lambda) = \mathcal{E}_0(\lambda)I_k$  at  $\lambda = \lambda_0$  and then multiplying by  $v_0^\top$  from the left and by  $v_0$  from the right.  $\blacksquare$

**3.2. Application to general first order differential operators.** In this section we apply the previous results to general first order differential operators with matrices that depend holomorphically on the eigenvalue parameter. We consider a first order  $(d \times d)$  matrix differential equation,

$$y' = A(\lambda, x)y, \quad x \in \mathbb{R}, \quad \lambda \in \Omega, \quad (3.34)$$

and the respective pencil of first order differential operators

$$F(\lambda)y = -y' + A(\lambda, x)y, \quad x \in \mathbb{R}, \quad \lambda \in \Omega, \quad (3.35)$$

and impose the following assumptions.

**Hypothesis 3.14.** (i) Assume that the mapping

$$\Omega \ni \lambda \mapsto A(\lambda, \cdot) - B(\cdot) \in L^\infty(\mathbb{R}, \mathbb{C}^{d,d}) \quad \text{is holomorphic} \quad (3.36)$$

for some matrix valued function  $B(\cdot)$  such that either  $B(\cdot) \in L^1(\mathbb{R}, \mathbb{C}^{d,d})$  or  $B(\cdot)$  is bounded continuous with  $\lim_{x \rightarrow \pm\infty} B(x) = 0$ .

(ii) Assume that the differential equation (3.34) has for all  $\lambda \in \Omega$  exponential dichotomy on  $\mathbb{R}_+$  with projections  $P_+(\lambda, x)$ ,  $x \geq 0$ , of rank  $k$  and on  $\mathbb{R}_-$  with projections  $P_-(\lambda, x)$ ,  $x \leq 0$ , of the same rank  $k$ , i.e. for every  $\lambda \in \Omega$  there exist  $C, \alpha > 0$  so that the usual dichotomy estimates in (3.41) below hold.

(iii) Assume that (3.34) has exponential dichotomy on  $\mathbb{R}$  for some  $\lambda \in \Omega$ .

Hypothesis 3.14(i) yields that the differential operators  $F(\lambda)$  are defined on the  $\lambda$ -independent domain, cf. [DL, Lem.2.8], [CL, Ch.3,4], given by

$$\mathcal{H} = \{y \in L^2(\mathbb{R}, \mathbb{C}^d) : y \in AC_{\text{loc}}(\mathbb{R}, \mathbb{C}^d), -y' + B(\cdot)y \in L^2(\mathbb{R}, \mathbb{C}^d)\}. \quad (3.37)$$

Moreover,  $F(\lambda)$  is a bounded operator from  $\mathcal{H}$  into the space

$$\mathcal{K} = L^2(\mathbb{R}, \mathbb{C}^d), \quad (3.38)$$

when  $\mathcal{H}$  is equipped with the graph norm

$$\|y\|_{\mathcal{H}}^2 = \|y\|_{L^2}^2 + \|-y' + B(\cdot)y\|_{L^2}^2.$$

If  $B(\cdot)$  is bounded, then  $\mathcal{H} = H^1(\mathbb{R}, \mathbb{C}^d)$ , the Sobolev space. The completeness of  $\mathcal{H}$  follows from Lemma A.1 saying that  $\mathcal{H}$  is embedded in the space of continuous functions vanishing at  $\pm\infty$ , see also [DL, Lem.2.8].

We note in passing that the differential operator  $F(\lambda)$ , considered as an unbounded operator in  $L^2(\mathbb{R}, \mathbb{C}^d)$  with the domain  $\mathcal{H}$ , generates a strongly continuous semigroup  $\{T^t\}_{t \geq 0}$ , called the evolution semigroup, see [CL], defined by the formula  $(T^t y)(x) = S(x, x-t, \lambda)y(x-t)$ ,  $x \in \mathbb{R}$ ,  $t \geq 0$ . Here and below we denote by  $S(x, \xi, \lambda)$ ,  $x, \xi \in \mathbb{R}$ , the propagator (solution operator) of the differential equation (3.34).

*Remarks 3.15.* (a) Hypothesis 3.14 (ii) holds if and only if the operator  $F(\lambda)$  is Fredholm and its Fredholm index is equal to zero for all  $\lambda \in \Omega$ . This is Palmer's Theorem, see [P84, P88], and also [BG], [Sa, Thm.3.2], [SaS1, Thm.2.6] and [LT, LP, SaS8] for its discussions and generalizations. In particular, the operator  $F(\lambda)$  is invertible if and only if (3.34) has an exponential dichotomy on the entire line  $\mathbb{R}$ .

(b) Hypothesis 3.14 (iii) implies that  $\rho(F) \neq \emptyset$ , cf. [BL, SaS8], [CL, Thm.3.17], and thus Hypothesis 3.14 yields Hypothesis 2.1 for pencil (3.35).

(c) Since the coefficient of the differential equation (3.34) is holomorphic, the dichotomy projections  $P_{\pm}(\cdot, x)$ ,  $x \in \mathbb{R}_{\pm}$ , are holomorphic, see, e.g., [BL], [DL, Lem.A1], [SaS0, Thm.1] and further references therein.  $\diamond$

*Example 3.16.* (Perturbations) A typical case where Hypotheses 3.14 are met occurs with operators of perturbation form

$$[F(\lambda)y](x) = -y'(x) + (A_0(\lambda, x) + B(x))y(x), \quad x \in \mathbb{R},$$

if the unperturbed operator  $F_0(\lambda)y = -y' + A_0(\lambda, x)y$  has an exponential dichotomy on  $\mathbb{R}$  for all  $\lambda \in \Omega$  and with projections of rank  $k$ . Then the assumption  $B \in L^1(\mathbb{R}, \mathbb{C}^{d,d})$  guarantees that the exponential dichotomies on half lines hold for the perturbed operator (see, e.g., [Co, Prop.4.1], [GLM, Lem.2.13]). This case applies to the Schrödinger equation in Section 5 below.  $\diamond$

A more specific situation occurs when one linearizes a one-dimensional PDE about a traveling front and re-writes the respective eigenvalue problem as the first order ODE system (3.34). Then its coefficient can be assumed to stabilize at  $\pm\infty$ .

*Example 3.17.* (Traveling Fronts) Consider the case of piecewise constant matrices

$$A_{\text{pc}}(\lambda, x) = A_+(\lambda) \text{ for } x \geq 0 \text{ and } A_{\text{pc}}(\lambda, x) = A_-(\lambda) \text{ for } x < 0, \quad (3.39)$$

where the matrices  $A_{\pm}(\lambda)$  satisfy the following properties:

- (a)  $A_{\pm}(\lambda)$  analytically depend on  $\lambda \in \Omega$ ,
- (b)  $A_{\pm}(\lambda)$  have no purely imaginary eigenvalues,
- (c)  $\text{rank } P_{A_+(\lambda)} = \text{rank } P_{A_-(\lambda)}$  for the Riesz projections  $P_{A_{\pm}(\lambda)}$  corresponding to the part of the spectrum of  $A_{\pm}(\lambda)$  located in the left half plane.

Under these conditions the (unperturbed) differential equation  $y' = A_{\text{pc}}(\lambda, x)y$  has exponential dichotomy on  $\mathbb{R}_{\pm}$  and is Fredholm of index zero by Palmer's result cited in Remark 3.15(a) above. This corresponds to fact that  $\lambda$  does not belong to the essential spectrum of the underlying differential operator that appears when one linearizes the PDE about the front, cf. [Sa]. Again, as in the previous example the perturbed differential equation  $y' = (A_{\text{pc}}(\lambda, x) + B(x))y$  inherits the dichotomies on half lines if either  $B \in L^1(\mathbb{R}, \mathbb{C}^{d,d})$  or  $B(\cdot)$  is bounded continuous with  $\lim_{x \rightarrow \pm\infty} B(x) = 0$ . In both cases we are in the system class described in Hypothesis 3.14. We cite [BL, LT, SaS8] for further references.  $\diamond$

As described in Section 2, we consider an inhomogenous equation

$$F(\lambda)y(\lambda) = \hat{v} \in L^2. \quad (3.40)$$

As above, let  $S(x, \xi, \lambda)$ ,  $x, \xi \in \mathbb{R}$ ,  $\lambda \in \Omega$  denote the solution operator of  $F(\lambda)$  (the propagator of the differential equation (3.34)), and let  $P_{\pm}(\lambda, x)$  be the dichotomy projections for (3.34) on  $\mathbb{R}_+$ ,  $\mathbb{R}_-$  from Hypothesis 3.14(ii). Then we have for some  $C, \alpha > 0$ ,

$$\begin{aligned} S(x, \xi, \lambda)P_{\pm}(\lambda, \xi) &= P_{\pm}(\lambda, x)S(x, \xi, \lambda) \quad \text{for } x, \xi \in \mathbb{R}_{\pm}, \\ \|S(x, \xi, \lambda)P_{\pm}(\lambda, \xi)\| &\leq Ce^{-\alpha(x-\xi)}, \quad \text{for } \xi \leq x, x, \xi \in \mathbb{R}_{\pm}, \\ \|S(x, \xi, \lambda)(I - P_{\pm}(\lambda, \xi))\| &\leq Ce^{-\alpha(\xi-x)}, \quad \text{for } x \leq \xi, x, \xi \in \mathbb{R}_{\pm}. \end{aligned} \quad (3.41)$$

Let  $\mathcal{E}$  be the Evans function from Theorem 3.9 (i) with respect to the equivalence classes  $[P(\lambda)], [Q(\lambda)]$  for any choice of holomorphic functions  $P, Q$  such that  $\mathcal{R}(P(\lambda)) = \mathcal{R}(P_+(\lambda, 0))$  and  $\mathcal{R}(Q(\lambda)) = \mathcal{N}(P_-(\lambda, 0))$ , i.e.

$$\mathcal{E}(\lambda) \in \mathcal{D}([P(\lambda)], [Q(\lambda)]), \quad \lambda \in \Omega. \quad (3.42)$$

Alternatively, let  $\mathcal{E}_0$  be the normalized Evans function from Theorem 3.9 (v) with respect to the subspaces  $U(\lambda) = \mathcal{R}(P_+(\lambda, 0))$  and  $V(\lambda) = \mathcal{N}(P_-(\lambda, 0))$  in  $\mathbb{C}^d$ , i.e.

$$\mathcal{E}_0(\lambda) \in \mathcal{D}(U(\lambda), V(\lambda)), \quad \lambda \in \Omega \setminus \Lambda(U, V). \quad (3.43)$$

We solve (3.40) for  $\lambda \in \rho(F)$  in a standard way by using Green's operators

$$\begin{aligned} (\mathcal{G}_+(\lambda)\hat{v})(x) &= \int_0^\infty G_+(x, \xi, \lambda)\hat{v}(\xi)d\xi, \quad x \geq 0, \\ (\mathcal{G}_-(\lambda)\hat{v})(x) &= \int_{-\infty}^0 G_-(x, \xi, \lambda)\hat{v}(\xi)d\xi, \quad x \leq 0, \end{aligned} \quad (3.44)$$

acting on functions  $\hat{v} : \mathbb{R} \rightarrow \mathbb{C}^d$ , with the kernels given by

$$\begin{aligned} G_+(x, \xi, \lambda) &= \begin{cases} S(x, \xi, \lambda)P_+(\lambda, \xi) & 0 \leq \xi \leq x, \\ S(x, \xi, \lambda)(P_+(\lambda, \xi) - I) & 0 \leq x < \xi, \end{cases} \\ G_-(x, \xi, \lambda) &= \begin{cases} S(x, \xi, \lambda)P_-(\lambda, \xi) & \xi \leq x \leq 0, \\ S(x, \xi, \lambda)(P_-(\lambda, \xi) - I) & x < \xi \leq 0. \end{cases} \end{aligned} \quad (3.45)$$

Due to the exponential dichotomies the operators  $\mathcal{G}_+, \mathcal{G}_-$  have uniform bounds in all spaces  $L^p(\mathbb{R}, \mathbb{C}^d)$ ,  $1 \leq p \leq \infty$ , see, e.g., [Co] or [CL, Sec.4.2]. The following piecewise defined function gives the general solution of (3.40) on both half lines (we write  $\hat{v}_+ = \hat{v}|_{\mathbb{R}_+}$ ,  $\hat{v}_- = \hat{v}|_{\mathbb{R}_-}$  for short),

$$y(\lambda, x) = \begin{cases} S(x, 0, \lambda)\eta_+(\lambda) + (\mathcal{G}_+(\lambda)\hat{v}_+)(x), & x \geq 0, \\ -S(x, 0, \lambda)\eta_-(\lambda) + (\mathcal{G}_-(\lambda)\hat{v}_-)(x), & x < 0, \end{cases} \quad (3.46)$$

where  $\eta_+(\lambda) \in U(\lambda)$  and  $\eta_-(\lambda) \in V(\lambda)$  are arbitrary. The function defined in (3.46) is a solution of the differential equation (3.34) if the left and right limits of  $y(\lambda)$  at zero coincide. Note that if  $y_{\pm} \in AC_{\text{loc}}(\mathbb{R}_{\pm}, \mathbb{C}^d)$  then the function  $y$  defined by

$$y(x) = \begin{cases} y_+(x), & x \geq 0, \\ y_-(x), & x < 0, \end{cases}$$

is in  $AC_{\text{loc}}(\mathbb{R}, \mathbb{C}^d)$  if and only if  $y_+(0) = y_-(0)$ . Therefore we have to find vectors  $\eta_+(\lambda) \in U(\lambda)$  and  $\eta_-(\lambda) \in V(\lambda)$  such that

$$\eta_+(\lambda) + \eta_-(\lambda) = (\mathcal{G}_-(\lambda)\hat{v}_-)(0) - (\mathcal{G}_+(\lambda)\hat{v}_+)(0) =: [\hat{v}]_0.$$

By Theorem 3.11 the sought for vectors are given by

$$\eta_+(\lambda) = \frac{1}{\mathcal{E}(\lambda)}\mathcal{Y}_U(\lambda)[\hat{v}]_0, \quad \eta_-(\lambda) = \frac{1}{\mathcal{E}(\lambda)}\mathcal{Y}_V(\lambda)[\hat{v}]_0. \quad (3.47)$$

Inserting this into (3.46) gives the solution formula

$$y(\lambda, x) = \begin{cases} \frac{1}{\mathcal{E}_0(\lambda)}S(x, 0, \lambda)\mathcal{Y}_U(\lambda)[\hat{v}]_0 + (\mathcal{G}_+(\lambda)\hat{v}_+)(x), & x \geq 0, \\ -\frac{1}{\mathcal{E}_0(\lambda)}S(x, 0, \lambda)\mathcal{Y}_V(\lambda)[\hat{v}]_0 + (\mathcal{G}_-(\lambda)\hat{v}_-)(x), & x < 0. \end{cases} \quad (3.48)$$

The same formulas hold with  $\mathcal{E}$  replaced by  $\mathcal{E}_0$  provided  $\lambda \in \Omega \setminus \Lambda(U, V)$ . It is convenient to introduce the operators  $\mathcal{G}(\lambda)$  acting on functions  $v : \mathbb{R} \rightarrow \mathbb{C}^d$  by

$$(\mathcal{G}(\lambda)v)(x) = \begin{cases} (\mathcal{G}_+(\lambda)v|_{\mathbb{R}_+})(x), & x \geq 0 \\ (\mathcal{G}_-(\lambda)v|_{\mathbb{R}_-})(x), & x < 0, \end{cases} \quad (3.49)$$

and the matrix valued function  $G$  by

$$G(\lambda, x) = \begin{cases} S(x, 0, \lambda)\mathcal{Y}_U(\lambda), & x \geq 0, \\ -S(x, 0, \lambda)\mathcal{Y}_V(\lambda), & x < 0, \end{cases} \quad (3.50)$$

so that if  $v_0 \in \mathbb{C}^d$  is a given vector then  $G(\lambda, \cdot)v_0 : \mathbb{R} \rightarrow \mathbb{C}^d$ .

Using these notions in (3.48) and inserting them into (2.4) finally leads to the expression (3.51) below for  $E_{jk}(\lambda)$ ,  $j = 1, \dots, m$ ,  $k = 1, \dots, \ell$ . To formulate the result, recall that the operators  $F(\lambda)$  act from  $\mathcal{H}$  into  $\mathcal{K}$ , see (3.35), (3.37), (3.38). Also, note that  $\mathcal{H} \subset L^2(\mathbb{R}, \mathbb{C}^d) \subset \mathcal{H}'$  such that a function  $w \in L^2(\mathbb{R}, \mathbb{C}^d)$  defines on  $\mathcal{H}$  a linear functional by  $\langle w, v \rangle = \langle w, v \rangle_{\mathbb{R}}$ ; here and below for  $w, v \in L^2(\mathbb{R}, \mathbb{C}^d)$  we denote  $\langle w, v \rangle_{\mathbb{R}} = \int_{-\infty}^{\infty} w(x)^{\top} v(x) dx$ . Following Section 2, we now choose linearly independent functions  $\hat{v}_k \in L^2(\mathbb{R}, \mathbb{C}^d)$ ,  $k = 1, \dots, \ell$ , and linearly independent functions  $\hat{w}_j \in L^2(\mathbb{R}, \mathbb{C}^d)$ ,  $j = 1, \dots, m$ , viewed as elements of  $\mathcal{H}'$ . Thus, the discussion above can be recorded as follows.

**Theorem 3.18.** *Assume Hypotheses 3.14, and let all eigenvalues of the operator pencil  $F$  inside  $\Omega_0$  be simple. Then the matrix  $E(\lambda) \in \mathbb{C}^{m, \ell}$  from the contour method (2.4), (2.5) satisfies the following formula*

$$E_{jk}(\lambda) = \frac{1}{\mathcal{E}(\lambda)} \langle \hat{w}_j, G(\lambda, \cdot)[\hat{v}_k]_0 \rangle_{\mathbb{R}} + \langle \hat{w}_j, \mathcal{G}(\lambda)\hat{v}_k \rangle_{\mathbb{R}}, \quad \lambda \in \Omega, \quad (3.51)$$

where the vector  $[\hat{v}_k]_0 = (\mathcal{G}_-(\lambda)\hat{v}_k)(0-) - (\mathcal{G}_+(\lambda)\hat{v}_k)(0+) \in \mathbb{C}^d$  denotes a jump quantity at  $x = 0$ , and the operators  $\mathcal{G}, \mathcal{G}_\pm$  and the matrix valued function  $G$  are defined in (3.44), (3.49) and (3.50). For  $\lambda \in \Omega \setminus \Lambda(U, V)$  formula (3.51) holds with  $\mathcal{E}$  replaced by the normalized Evans function  $\mathcal{E}_0$ .

Formula (3.51) shows how to express the abstract terms in (3.1) through integral kernels and the Evans function (or the normalized Evans function) for first order systems.

To conclude this subsection, we summarize our results for the operator pencil (3.35). In particular, we apply Theorem 3.13 to recover the singular part of the function  $E_{jk}(\cdot)$  near a simple eigenvalue  $\lambda_n, n = 1, \dots, \varkappa$ . Recall that  $P(\lambda), Q(\lambda) \in \mathbb{H}(\Omega, \mathbb{C}^{d,k})$  are chosen as in (3.16) with  $\Pi_U(\lambda) = P_+(\lambda, 0)$  and  $\Pi_V(\lambda) = I - P_-(\lambda, 0)$ , and the Evans function is defined by  $\mathcal{E}(\lambda) = \det(P(\lambda)|Q(\lambda))$ .

**Theorem 3.19.** *Assume Hypotheses 3.14. The following assertions are equivalent.*

- (i)  $\lambda_0$  is a simple eigenvalue of the operator pencil  $F$  (3.35);
- (ii)  $\lambda_0$  is a simple root of the Evans function  $\mathcal{E}$ ;
- (iii)  $\dim \mathcal{N}(P(\lambda_0)|Q(\lambda_0)) = 1$ ;
- (iv)  $\dim(\mathcal{R}(P_+(\lambda_0, 0)) \cap \mathcal{N}(P_-(\lambda_0, 0))) = 1$ ;
- (v) There exists a unique up to a scalar multiple exponentially decaying at  $\pm\infty$  solution  $v$  of (3.34);
- (vi) There exists a unique up to a scalar multiple exponentially decaying at  $\pm\infty$  solution  $w$  of the adjoint to (3.34) equation  $(z^\top)' = -z^\top A(\lambda, x)^\top$ .

Moreover, if  $v(0)$  denotes the initial value of the exponentially decaying solution in (v), then  $v(0) \in \mathcal{R}(P_+(\lambda_0, 0)) \cap \mathcal{N}(P_-(\lambda_0, 0))$  if and only if

$$v(0) = P(\lambda_0)v_{0,1} = -Q(\lambda_0)v_{0,2} \quad (3.52)$$

for a vector  $v_0 = \begin{pmatrix} v_{0,1} \\ v_{0,2} \end{pmatrix}$  from  $\mathcal{N}(P(\lambda_0)|Q(\lambda_0))$ .

In addition, assume that all eigenvalues  $\lambda_n, n = 1, \dots, \varkappa$ , in  $\Omega_0$  are simple, let  $v_n, w_n^\top$  be the solutions described in assertions (v), (vi) above for each  $\lambda_n$ , and normalized as indicated in (2.8), let  $\{\hat{v}_k\}_{k=1}^\ell, \{\hat{w}_j\}_{j=1}^m$  be linearly independent functions in  $L^2(\mathbb{R}, \mathbb{C}^d)$  chosen as indicated in Section 2. Then the singular part of the function  $E_{jk}(\cdot)$  defined in (2.4) in the framework of the abstract Keldysh theorem (2.10) is given by the formula

$$E_{jk}^{\text{sing}}(\lambda) = \sum_{n=1}^{\varkappa} \frac{1}{\lambda - \lambda_n} \langle \hat{w}_j, v_n \rangle_{\mathbb{R}} \langle w_n, \hat{v}_k \rangle_{\mathbb{R}}, \quad |\lambda - \lambda_n| \ll 1, \quad n = 1, \dots, \varkappa. \quad (3.53)$$

*Proof.* The three assertions,  $\mathcal{E}(\lambda_0) = 0$ ,  $\mathcal{R}(P_+(\lambda_0, 0)) \cap \mathcal{N}(P_-(\lambda_0, 0)) \neq \{0\}$ , and  $\mathcal{N}(F(\lambda_0)) \neq \{0\}$ , are equivalent by Theorem 3.9 and by the dichotomy assumptions in Hypothesis 3.14. Moreover,  $\mathcal{R}(P_+(\lambda_0, 0)) \cap \mathcal{N}(P_-(\lambda_0, 0))$  and  $\mathcal{N}(F(\lambda_0))$  are isomorphic via the map  $v(0) \mapsto v(\cdot) = S(\cdot, 0, \lambda_0)v(0)$ . Thus to see the equivalence of the first four items in the theorem it suffices to show that the subspaces  $\mathcal{R}(P_+(\lambda_0, 0)) \cap \mathcal{N}(P_-(\lambda_0, 0))$  and  $\mathcal{N}(P(\lambda_0)|Q(\lambda_0))$  are isomorphic as indicated in (3.52). Let  $v_0 \in \mathcal{N}(P(\lambda_0)|Q(\lambda_0))$ . Since  $\mathcal{R}(P(\lambda_0)) = \mathcal{R}(P_+(\lambda_0, 0))$  and  $\mathcal{R}(Q(\lambda_0)) = \mathcal{N}(P_-(\lambda_0, 0))$ , we have  $P(\lambda_0)v_{0,1} \in \mathcal{R}(P_+(\lambda_0, 0))$  and  $Q(\lambda_0)v_{0,2} \in \mathcal{N}(P_-(\lambda_0, 0))$ , and by the choice of  $v_0$  we have  $P(\lambda_0)v_{0,1} = -Q(\lambda_0)v_{0,2}$ . Then  $v(0)$  from (3.52) belongs to  $\mathcal{R}(P_+(\lambda_0, 0)) \cap \mathcal{N}(P_-(\lambda_0, 0))$ . Conversely, if  $v(0) \in \mathcal{R}(P_+(\lambda_0, 0)) \cap \mathcal{N}(P_-(\lambda_0, 0))$  then  $v(0) = P(\lambda_0)v_{0,1}$  and  $v(0) = -Q(\lambda_0)v_{0,2}$  for

some  $v_{0,1} \in \mathbb{C}^k$ ,  $v_{0,2} \in \mathbb{C}^{d-k}$ . Letting  $v_0 = \begin{pmatrix} v_{0,1} \\ v_{0,2} \end{pmatrix}$  yields  $v_0 \in \mathcal{N}(P(\lambda_0)|Q(\lambda_0))$ , as required. To begin the proof of (vi), we remark that  $\dim \mathcal{N}(P(\lambda_0)|Q(\lambda_0))$  is equal to  $\dim \mathcal{N}((P(\lambda_0)|Q(\lambda_0))^\top)$ ; also, the following identities hold:

$$\begin{aligned} \mathcal{N}((P(\lambda_0)|Q(\lambda_0))^\top) &= \mathcal{R}((P(\lambda_0)|Q(\lambda_0)))^\perp = \mathcal{R}(P(\lambda_0))^\perp \cap \mathcal{R}(Q(\lambda_0))^\perp \\ &= \mathcal{R}(P_+(\lambda_0, 0))^\perp \cap \mathcal{N}(P_-(\lambda_0, 0))^\perp = \mathcal{N}(P_+(\lambda_0, 0)^\top) \cap \mathcal{R}(P_-(\lambda_0, 0)^\top) \\ &= \mathcal{R}(I - P_+(\lambda_0, 0)^\top) \cap \mathcal{N}(I - P_-(\lambda_0, 0)^\top). \end{aligned}$$

Since (3.34) has the exponential dichotomy  $P_\pm(\lambda_0, 0)$  if and only if the adjoint equation has the exponential dichotomy  $I - P_\pm(\lambda_0, 0)^\top$  (see, e.g., [BG, Lem.4.5], [DL, Lem.2.4], [Sa, Rem.3.4]), it follows that the subspaces  $\mathcal{N}((P(\lambda_0)|Q(\lambda_0))^\top)$  and  $\mathcal{N}(F(\lambda_0)^\top)$  are isomorphic via the map  $w(0) \mapsto w(\cdot)^\top$ , where  $w$  is the exponentially decaying solution of the adjoint equation.

It remains to show (3.53). For each  $n = 1, \dots, \varkappa$  let  $v_n(0)$  be the vector from (3.52), let  $v_n(x) = S(x, 0, \lambda_n)v_n(0)$  be the decaying on both  $\mathbb{R}_+$  and  $\mathbb{R}_-$  solution of (3.34), and let  $w_n(0)$  span  $\mathcal{N}((P(\lambda_n)|Q(\lambda_n))^\top)$ . We obtain from (3.30) that

$$\mathcal{Y}_U(\lambda_n) = \mathcal{E}'(\lambda_n)v_n(0)w_n(0)^\top \quad \text{and} \quad \mathcal{Y}_V(\lambda_n) = -\mathcal{E}'(\lambda_n)v_n(0)w_n(0)^\top,$$

and from (3.50) that  $G(\lambda_n, x) = \mathcal{E}'(\lambda_n)v_n(x)w_n(0)^\top$ . Using (3.51) and  $\mathcal{E}(\lambda) = \mathcal{E}'(\lambda_n)(\lambda - \lambda_n) + \mathcal{O}(|\lambda - \lambda_n|^2)$  yields the singular part

$$E_{jk}^{\text{sing}}(\lambda) = \frac{1}{\lambda - \lambda_n} \langle \hat{w}_j, v_n \rangle w_n(0)^\top [\hat{v}_k]_0, \quad |\lambda - \lambda_n| \ll 1. \quad (3.54)$$

Note that

$$\begin{aligned} w_n(0)^\top [\hat{v}_k]_0 &= w_n^\top(0) \left( \int_{-\infty}^0 S(0, \xi, \lambda) P_-(\lambda, \xi) \hat{v}_k(\xi) d\xi \right. \\ &\quad \left. - \int_0^\infty S(0, \xi, \lambda) (P_+(\lambda, \xi) - I) \hat{v}_k(\xi) d\xi \right). \end{aligned}$$

Next we observe that

$$w_n^\top(\xi) = \begin{cases} w_n^\top(0) S(0, \xi, \lambda) P_-(\lambda, \xi), & \xi \leq 0, \\ w_n^\top(0) S(0, \xi, \lambda) (I - P_+(\lambda, \xi)), & \xi > 0 \end{cases}$$

solves the adjoint equation of (3.34), is continuous at 0, and decays exponentially in both directions. Using this in (3.54) finally leads us back to the singular part determined by the abstract Keldysh theorem in (2.10), where the term  $\langle w_n, \hat{v}_k \rangle$  is understood as the integral  $\langle w_n, \hat{v}_k \rangle_{\mathbb{R}}$ .  $\blacksquare$

#### 4. CONVERGENCE OF EIGENVALUES FOR FINITE BOUNDARY VALUE PROBLEMS

In this section we provide error estimates of the eigenvalues obtained by the contour method in Section 2 when the boundary value problems (3.40) are solved approximately on a bounded interval. In the first step we analyze the error of the boundary value solutions themselves, and in the second step we discuss the implications for the contour method.

**4.1. Estimates of boundary value solutions.** Using the setting and notation from Section 3.2 we consider the all-line boundary value problem

$$F(\lambda)y(\lambda) = -y'(\lambda, \cdot) + A(\lambda, \cdot)y(\lambda, \cdot) = \widehat{v} \in L^2(\mathbb{R})$$

for various values of  $\lambda$  and  $\widehat{v}$ . Our main assumption is the following.

**Hypothesis 4.1.** *Let Hypotheses 3.14 hold and assume that the dichotomy exponent  $\alpha > 0$  in (3.41) is uniform for all  $\lambda \in \Omega$  and that the dichotomy projections  $P_{\pm}(\lambda, x)$  given in Hypothesis 3.14 (ii) are asymptotically constant, that is,  $\lim_{x \rightarrow \pm\infty} P_{\pm}(\lambda, x) = P_{\pm}(\lambda)$ .*

We approximate (3.40) by a sequence of boundary value problems on finite intervals  $J_N = [x_-^N, x_+^N]$ ,  $N \in \mathbb{N}$ , with  $-x_-^N, x_+^N \rightarrow \infty$  as  $N \rightarrow \infty$ :

$$F_N(\lambda)y := \begin{pmatrix} -y' + A(\lambda, \cdot)y \\ R_-(\lambda)y(x_-^N) + R_+(\lambda)y(x_+^N) \end{pmatrix} = \begin{pmatrix} \widehat{v}|_{J_N} \\ 0 \end{pmatrix} \in L^2(J_N) \times \mathbb{C}^d, \quad (4.1)$$

where  $R_-, R_+ \in \mathbb{H}(\Omega, \mathbb{C}^{d,d})$  are given matrix valued functions. We allow boundary conditions that are nonlinear in the eigenvalue parameter in order to cover so-called projection boundary conditions which lead to fast convergence of solutions of (4.1) as  $-x_-^N, x_+^N \rightarrow \infty$ , see (4.16) below.

We will show how (4.1) fits into the framework of Section 2.4 and apply Theorem 2.10 to obtain error estimates. Our approach is largely based on [BR] where the case of smooth coefficients and  $\text{dom } F(\lambda) = H^1(\mathbb{R}, \mathbb{C}^d)$ , the Sobolev space, was analyzed. For any interval  $J \subseteq \mathbb{R}$  introduce the Banach space

$$\mathcal{H}_J = \{y \in L^2(J, \mathbb{C}^d) : y \in AC_{\text{loc}}(J, \mathbb{C}^d), -y' + By \in L^2(J, \mathbb{C}^d)\} \quad (4.2)$$

with norm  $\|y\|_{\mathcal{H}_J}^2 = \|y\|_{L^2(J)}^2 + \|-y' + By\|_{L^2(J)}^2$ . Note that  $\mathcal{H}_{\mathbb{R}}$  agrees with  $\mathcal{H}$  from (3.37). Using Lemma A.2 it is easy to see that the space  $\mathcal{H}_J$  is complete in this norm (for later reference we collect further properties of  $\mathcal{H}_J$  in the Appendix).

The spaces  $\mathcal{H}_N$  and  $\mathcal{K}_N$  from Section 2.4 are defined by  $\mathcal{H}_N = \mathcal{H}_{J_N}$  and  $\mathcal{K}_N = L^2(J_N, \mathbb{C}^d) \times \mathbb{C}^d$  with norms  $\|y\|_{\mathcal{H}_N}$  and  $\|(v, r)\|_{\mathcal{K}_N}^2 = \|v\|_{L^2}^2 + |r|^2$ . The spaces  $\mathcal{H}$ ,  $\mathcal{K}$  are taken as in (3.37), (3.38) and mapped into  $\mathcal{H}_N, \mathcal{K}_N$  by

$$p_N y = y|_{J_N} \quad \text{and} \quad q_N v = (v|_{J_N}, 0). \quad (4.3)$$

Obviously these mappings are linear, bounded uniformly in  $N$  and satisfy condition (D1) from Section 2.4.

By Hypothesis 3.14 (i),(ii) we have  $F \in \mathbb{H}(\Omega, \mathcal{F}(\mathcal{H}, \mathcal{K}))$  since  $F(\lambda)$  is Fredholm for each  $\lambda$  in  $\Omega$  and the map  $\Omega \ni \lambda \mapsto A(\lambda, \cdot) - B(\cdot) \in L^\infty(\mathbb{R}, \mathbb{C}^{d,d})$  is holomorphic. Moreover,  $\rho(F) \neq \emptyset$  by Hypothesis 3.14(iii), and (D2) follows from the next lemma.

**Lemma 4.2.** *Under the above assumptions on  $A, R_-, R_+$  the operators  $F_N$  are in  $\mathbb{H}(\Omega, \mathcal{F}(\mathcal{H}_N, \mathcal{K}_N))$  and  $\sup_{N \in \mathbb{N}} \sup_{\lambda \in \mathcal{C}} \|F_N(\lambda)\| < \infty$  for every compact set  $\mathcal{C} \subset \Omega$ .*

*Proof.* Let  $y \in \mathcal{H}_N$  and  $\lambda \in \Omega$ . Then, by (3.36) and Lemma A.2,

$$\begin{aligned} \|F_N(\lambda)y\|_{\mathcal{K}_N}^2 &= \|-y' + A(\lambda, \cdot)y\|_{L^2(J_N)}^2 + |R_-(\lambda)y(x_-^N) + R_+(\lambda)y(x_+^N)|^2 \\ &\leq 2(\|y\|_{\mathcal{H}_N}^2 + \|(A(\lambda, \cdot) - B(\cdot))y\|_{L^2(J_N)}^2) + 2(\|R_-(\lambda)\|^2 + \|R_+(\lambda)\|^2)\|y\|_{L^\infty}^2 \\ &\leq c(\|A(\lambda, \cdot) - B(\cdot)\|_{L^\infty}^2 + \|R_-(\lambda)\|^2 + \|R_+(\lambda)\|^2)\|y\|_{\mathcal{H}_N}^2. \end{aligned}$$

From the holomorphy of  $A, R_-, R_+$  we obtain uniform bounds for  $\|F_N(\lambda)\|$  on compact sets  $\mathcal{C} \subset \Omega$  as well as holomorphy of  $\lambda \mapsto F_N(\lambda)$  for all  $N \in \mathbb{N}$ . Finally, the Fredholm property is a well-known fact for finite boundary value problems. ■

For the application of Theorem 2.10 it remains to verify (D3). Let  $V_-^s(\lambda)$  be a basis of the range of  $P_-(\lambda)$  and let  $V_+^u(\lambda)$  be a basis of the kernel of  $P_+(\lambda)$ .

**Proposition 4.3.** *Let Hypothesis 4.1 hold and assume that matrices  $R_\pm(\lambda)$  for all  $\lambda \in \Omega$  satisfy*

$$\det((R_-(\lambda)V_-^s(\lambda)|R_+(\lambda)V_+^u(\lambda))) \neq 0. \quad (4.4)$$

*Then  $F_N(\lambda)$  converges regularly to  $F(\lambda)$  for all  $\lambda \in \Omega$ .*

*Proof.* Let  $y \in \mathcal{H}$ ,  $\lambda \in \Omega$ . Then

$$\|F_N(\lambda)p_N y - q_N F(\lambda)y\|_{\mathcal{K}_N}^2 \leq 2(\|R_-(\lambda)\|^2 + \|R_+(\lambda)\|^2)(|y(x_-^N)|^2 + |y(x_+^N)|^2), \quad (4.5)$$

where the right-hand side converges to 0 as  $N \rightarrow \infty$  by Lemma A.1. This proves part (a) of (D3).

Let  $\lambda \in \Omega$  be fixed. Consider a subsequence  $y_N \in \mathcal{H}_N$ ,  $N \in \mathbb{N}'$ , with bounded  $\|y_N\|_{\mathcal{H}_N}$  and assume there is  $v \in \mathcal{K}$  with  $\lim_{N \rightarrow \infty} \|F_N(\lambda)y_N - q_N v\|_{\mathcal{K}_N} = 0$ . Set  $(v_N, r_N) := F_N(\lambda)y_N \in \mathcal{K}_N$  and note that  $y_N$  can be written similarly to (3.46):

$$y_N(x) = \begin{cases} G_+(x, 0, \lambda)y_N(0) - G_+(x, x_+^N, \lambda)y_N(x_+^N) + [\mathcal{G}_+^N(\lambda)v_N](x), & x \geq 0, \\ -G_-(x, 0, \lambda)y_N(0) + G_-(x, x_-^N, \lambda)y_N(x_-^N) + [\mathcal{G}_-^N(\lambda)v_N](x), & x \leq 0, \end{cases} \quad (4.6)$$

where  $G_\pm$  are defined in (3.45) and, cf. (3.44),

$$\begin{aligned} (\mathcal{G}_+^N(\lambda)v_N)(x) &= \int_0^{x_+^N} G_+(x, \xi, \lambda)v_N(\xi) d\xi, \quad x_+^N \geq x \geq 0, \\ (\mathcal{G}_-^N(\lambda)v_N)(x) &= \int_{x_-^N}^0 G_-(x, \xi, \lambda)v_N(\xi) d\xi, \quad x_-^N \leq x \leq 0. \end{aligned}$$

By Lemma A.2 the boundedness of  $\|y_N\|_{\mathcal{H}_N}$  implies boundedness and thus compactness of the sequence  $(y_N(0))_{N \in \mathbb{N}'} \subset \mathbb{C}^d$ , i.e.  $\lim_{\mathbb{N}'' \ni N \rightarrow \infty} y_N(0) = y_0$  for some subsequence  $\mathbb{N}'' \subset \mathbb{N}'$  and some  $y_0 \in \mathbb{C}^d$ . Let

$$y(x) = \begin{cases} y_+(x) = G_+(x, 0, \lambda)y_0 + [\mathcal{G}_+(\lambda)v|_{\mathbb{R}_+}](x), & x \geq 0, \\ y_-(x) = -G_-(x, 0, \lambda)y_0 + [\mathcal{G}_-(\lambda)v|_{\mathbb{R}_-}](x), & x < 0. \end{cases} \quad (4.7)$$

Now we follow verbatim the proof of [BR, Thm.2.1] until [BR, (2.11)] to conclude

$$\lim_{\mathbb{N}'' \ni N \rightarrow \infty} \|y_N - y|_{J_N}\|_{L^2(J_N)}^2 = 0. \quad (4.8)$$

Note that this step is crucial. It uses the determinant condition (4.4) as well as the representation (4.6) and the exponential dichotomies. It remains to prove  $\|y_N - p_N y\|_{\mathcal{H}_N} \rightarrow 0$  as  $\mathbb{N}'' \ni N \rightarrow \infty$ , for which the arguments in [BR] do no longer apply.

By construction  $y_\pm \in AC_{\text{loc}}(\mathbb{R}_\pm, \mathbb{C}^d) \cap L^2(\mathbb{R}_\pm, \mathbb{C}^d)$  and

$$-y'_\pm + A(\lambda, \cdot)y_\pm = v|_{\mathbb{R}_\pm} \quad (4.9)$$

holds in  $L^2(\mathbb{R}_\pm, \mathbb{C}^d)$ . Therefore, the function defined by

$$z(x) = \begin{cases} -y'_+ + B(x)y_+(x), & x \geq 0, \\ -y'_- + B(x)y_-(x), & x < 0, \end{cases} \quad (4.10)$$

is in  $L^2(\mathbb{R}, \mathbb{C}^d)$  and satisfies  $z = v + (B(\cdot) - A(\lambda, \cdot))y$ . Using this we obtain

$$\|-y'_N + B(\cdot)y_N - z|_{J_N}\|_{L^2(J_N)} = \|-y'_N + B(\cdot)y_N - (v + (B(\cdot) - A(\lambda, \cdot))y)|_{J_N}\|_{L^2(J_N)}$$



$$\leq \|F_N(\lambda)y_N - q_N v\|_{\mathcal{K}_N} + \|A(\lambda, \cdot) - B(\cdot)\|_{L^\infty} \|y_N - y|_{J_N}\|_{L^2(J_N)}, \quad (4.11)$$

where the right-hand side converges to zero as  $\mathbb{N}'' \ni N \rightarrow \infty$  by (4.8) and our assumption. Without loss of generality we may assume  $J_N \supset [-1, 1]$  for all  $N \in \mathbb{N}''$ . Repeating the estimate (4.11) with  $[-1, 1]$  instead of  $J_N$  shows that  $(y_N|_{[-1, 1]})_{N \in \mathbb{N}''}$  is a Cauchy sequence in  $\mathcal{H}_{[-1, 1]}$ . Therefore, its limit, which coincides with  $y|_{[-1, 1]}$ , is an element of  $AC([-1, 1])$ . This allows us to conclude  $z = -y' + B(\cdot)y$  from (4.10), so that equations (4.8) and (4.11) prove our final assertion.  $\blacksquare$

The above results show that the abstract convergence result, Theorem 2.10, applies:

**Theorem 4.4.** *Let the assumptions of Proposition 4.3 hold. Then for any compact set  $\mathcal{C} \subset \rho(F) \cap \Omega$  and any  $\hat{v} \in \mathcal{K}$  there is  $N_0 \in \mathbb{N}$  such that for all  $N \geq N_0$  and  $\lambda \in \mathcal{C}$  the boundary value problem (4.1) has a unique solution  $y_N(\lambda, \cdot) \in \mathcal{H}_N$ . Furthermore, for some constant  $C$ , independent of  $\hat{v}$ ,*

$$\sup_{\lambda \in \mathcal{C}} \|y_N(\lambda, \cdot) - p_N y(\lambda, \cdot)\|_{\mathcal{H}_N} \leq C \sup_{\lambda \in \mathcal{C}} |R_-(\lambda)y(\lambda, x_-^N) + R_+(\lambda)y(\lambda, x_+^N)|, \quad (4.12)$$

where  $y(\lambda, \cdot) \in \mathcal{H}$  solves (3.40).

From the well known decay  $|y(\lambda, x_\pm^N)| \rightarrow 0$  as  $N \rightarrow \infty$  (e.g. see the proof of [BL, Thm.3.2]) estimate (4.12) implies that the solutions  $y_N(\lambda, \cdot)$  of the finite interval problems converge uniformly in  $\lambda \in \mathcal{C}$  to the solution of the problem on the line.

We will now concentrate on the differential equation (3.34) with the coefficient of the special perturbative structure which appears in the case of traveling fronts with asymptotic hyperbolic rest states, see Example 3.17. Specifically, let us consider a first order operator of the form (3.35) where

$$A(\lambda, x) = A_{\text{pc}}(\lambda, x) + B(x), \quad x \in \mathbb{R}, \quad (4.13)$$

with  $A_{\text{pc}}(\lambda, x)$  defined in (3.39). We impose the following assumptions.

**Hypothesis 4.5.** *The differential equation (3.34) with  $A(\lambda, x)$  from (4.13) satisfies Hypothesis 3.14 and Hypothesis 4.1 with the uniform exponential estimate,*

$$\|P_\pm(\lambda, x) - P_\pm(\lambda)\| \leq ce^{-\alpha|x|}, \quad x \in \mathbb{R}_\pm, \quad (4.14)$$

for all  $\lambda \in \Omega$ , where  $\alpha$  is the exponent from (3.41) of the exponential dichotomy on  $\mathbb{R}_\pm$  for (3.34). The projections  $P_\pm(\lambda)$  depend analytically on  $\lambda \in \Omega$ .

A typical situation where Hypothesis 4.5 is satisfied, occurs when the matrix-valued function  $A(\lambda, \cdot) = A_{\text{pc}}(\lambda, \cdot) + B(\cdot)$  is continuous, assumptions (a) – (c) in Example 3.17 hold, and there is  $c > 0$  such that for all  $\lambda \in \Omega$ ,

$$\|A(\lambda, x) - A_-(\lambda)\| \leq ce^{-\alpha|x|}, \quad x \leq 0, \quad \|A(\lambda, x) - A_+(\lambda)\| \leq ce^{-\alpha|x|}, \quad x \geq 0, \quad (4.15)$$

where  $\alpha$  is the exponent of exponential dichotomy on  $\mathbb{R}_\pm$  for the constant coefficient equations  $y' = A_\pm(\lambda)y$ . Then the differential equation (3.34) with  $A(\lambda, x)$  as in (4.13) has exponential dichotomy on  $\mathbb{R}_\pm$  for all  $\lambda \in \Omega$  and the roughness theorem [BL, Thm.A.3] implies (4.14). Thus Hypothesis 4.5 is satisfied provided (3.34) has an exponential dichotomy on  $\mathbb{R}$  for at least one  $\lambda \in \Omega$ .

Under Hypothesis 4.5, projection boundary conditions in (4.1) are a suitable choice, because they always satisfy (4.4) by construction. For convenience, we recall the definition of the projection boundary conditions, see [B90] for more details. Since the limits  $P_\pm(\lambda)$  depend analytically on  $\lambda$ , there are analytic bases  $V_\pm^s(\lambda)$ ,

respectively,  $V_{\pm}^u(\lambda)$  of  $\mathcal{R}(P_{\pm}(\lambda))$ , respectively,  $\mathcal{N}(P_{\pm}(\lambda))$  (e.g. [K, Sec.II.1.4]). Let us split the inverse matrix composed as follows:

$$(V_{\pm}^s(\lambda)|V_{\pm}^u(\lambda))^{-1} = \begin{pmatrix} L_{\pm}^s(\lambda) \\ L_{\pm}^u(\lambda) \end{pmatrix}, \quad L_{\pm}^s(\lambda) \in \mathbb{C}^{k,d}, \quad L_{\pm}^u(\lambda) \in \mathbb{C}^{d-k,d}.$$

The projection boundary conditions are then given by the boundary matrices

$$R_{-}(\lambda) := \begin{pmatrix} L_{-}^s(\lambda) \\ 0_{(d-k) \times d} \end{pmatrix} \in \mathbb{C}^{d,d}, \quad R_{+}(\lambda) := \begin{pmatrix} 0_{k \times d} \\ L_{+}^u(\lambda) \end{pmatrix} \in \mathbb{C}^{d,d}. \quad (4.16)$$

By construction  $(R_{-}(\lambda)V_{-}^s(\lambda)|R_{+}(\lambda)V_{+}^u(\lambda)) = I_d$ , i.e. (4.4) is satisfied, and

$$R_{-}(\lambda)(I - P_{-}(\lambda)) = 0, \quad R_{+}(\lambda)P_{+}(\lambda) = 0. \quad (4.17)$$

We now use Theorem 4.4 to establish our main convergence result for the matrix in (2.4) and the integrals in (2.5) when the underlying boundary value problems are solved on a finite interval. We denote by  $\mathcal{M}_b^c = \mathcal{M}_b^c(\mathbb{R}, \mathbb{C}^d)$  the set of finite, compactly supported,  $\mathbb{C}^d$  valued Radon measures on  $\mathbb{R}$ . By Riesz's Theorem, e.g. [F, Thm.7.17], and Lemma A.1,  $\mathcal{M}_b^c \subset \mathcal{H}'$  for  $\mathcal{H}$  from (3.37). If  $\hat{w} \in \mathcal{H}'$  is given by  $\mu \in \mathcal{M}_b^c$ , we write  $\langle \hat{w}, v \rangle = \int_{\mathbb{R}} v^{\top}(x) d\mu$  for  $v \in \mathcal{H}$ . We approximate  $\hat{w} = \mu \in \mathcal{M}_b^c \subset \mathcal{H}'$  on a finite interval  $J$  by its trace  $\hat{w}|_J = \mu|_J$  defined through  $\langle \hat{w}|_J, v \rangle = \int_J v^{\top}(x) d\mu$  for all  $v \in \mathcal{H}_J$ . Obviously,

$$\langle \hat{w}|_J, p_J v \rangle - \langle \hat{w}, v \rangle = \int_{\mathbb{R} \setminus J} v^{\top}(x) d\mu = 0, \quad \text{if } J \supset \text{supp}(\mu) \text{ and } v \in \mathcal{H}. \quad (4.18)$$

*Example 4.6.* Two standard examples for  $\hat{w} \in \mathcal{H}'$ , given as  $\mu \in \mathcal{M}_b^c$ :

- (1) If  $\mu = e_i \delta_{x_0}$  for some  $x_0 \in \mathbb{R}$ ,  $i \in \{1, \dots, d\}$ , then  $\langle \hat{w}, v \rangle = \int v^{\top}(x) d\mu = v(x_0)^{\top} e_i = v_i(x_0)$  is the  $i$ 'th component of  $v$  evaluated at  $x_0$ .
- (2) If  $\mu$  has density  $f \in L_{\text{loc}}^1(\mathbb{R}, \mathbb{C}^d)$  with respect to the Lebesgue measure, then  $\langle \hat{w}, v \rangle = \int_{\mathbb{R}} v^{\top}(x) f(x) dx$ .  $\diamond$

After these preliminaries we define and estimate approximations  $E^N(\lambda)$  of the function  $E(\lambda)$  from (2.4) by solving finite interval boundary value problems.

**Theorem 4.7.** *Let  $F$  from (3.35) satisfy Hypothesis 4.1, and let condition (4.4) hold for boundary matrices  $R_{\pm}(\lambda)$ . Moreover, let  $\Gamma \subset \rho(F) \cap \Omega$  be a contour and assume linearly independent elements  $\hat{w}_j \in \mathcal{H}'$ ,  $j = 1, \dots, m$ , defined by  $\mu_j \in \mathcal{M}_b^c$ , and linearly independent functions  $\hat{v}_k \in \mathcal{K}$ ,  $k = 1, \dots, \ell$  which are bounded and have compact support. Then the finite interval approximation  $E^N(\lambda)$  of  $E(\lambda)$ , defined by (cf. (4.1))*

$$F_N(\lambda) y_k^N(\lambda) = (\hat{v}_k|_{J_N}, 0), \quad k = 1, \dots, \ell,$$

$$E^N(\lambda)_{jk} = \langle \hat{w}_j|_{J_N}, y_k^N(\lambda) \rangle = \int_{J_N} y_k^N(\lambda)^{\top} d\mu_j, \quad j = 1, \dots, m,$$

satisfies

$$\sup_{\lambda \in \Gamma} \|E^N(\lambda) - E(\lambda)\| \leq c e^{-\alpha \min\{-x_-^N, x_+^N\}}, \quad (4.19)$$

with  $\alpha$  being the dichotomy exponent in  $\Omega$  from Hypothesis 4.1 and  $c$  a uniform constant.

If  $F$  additionally satisfies Hypothesis 4.5 and  $F_N$  is given with the projection boundary conditions defined via (4.16), then (4.19) improves to

$$\sup_{\lambda \in \Gamma} \|E^N(\lambda) - E(\lambda)\| \leq ce^{-2\alpha \min\{-x_-^N, x_+^N\}}. \quad (4.20)$$

As a corollary we obtain estimates for the approximate matrices (cf. (2.5))

$$D_j^N = \frac{1}{2\pi i} \int_{\Gamma} \lambda^j E^N(\lambda) d\lambda, \quad j = 0, 1. \quad (4.21)$$

**Corollary 4.8.** *Under the assumptions of Theorem 4.7 the following estimates hold*

$$\|D_0 - D_0^N\| \leq ce^{-\alpha \min\{-x_-^N, x_+^N\}}, \quad \|D_1 - D_1^N\| \leq ce^{-\alpha \min\{-x_-^N, x_+^N\}}. \quad (4.22)$$

In the case of projection boundary conditions the constant  $\alpha$  improves to  $2\alpha$ .

*Proof of Theorem 4.7.* Throughout the proof,  $c$  is a generic constant. For all  $\lambda \in \Gamma \subset \rho(F)$  the differential equation (3.34) has an exponential dichotomy on  $\mathbb{R}$  with a uniform exponent  $\alpha$  and projections  $P(\lambda, x) = P_{\pm}(\lambda, x)$  that depend holomorphically on  $\lambda$  and satisfy  $\lim_{x \rightarrow \pm\infty} P(\lambda, x) = P_{\pm}(\lambda)$ , see [BL, Thm. A.5]. With these projections the Green's function reads

$$G(x, \xi, \lambda) = \begin{cases} S(x, \xi, \lambda)P(\lambda, \xi), & x \geq \xi, \\ S(x, \xi, \lambda)(P(\lambda, \xi) - I), & x < \xi, \end{cases}$$

and the solution  $y_k(\lambda)$  of  $F(\lambda)y_k(\lambda) = \widehat{v}_k$  is given by (see [BL, Thm.A.1])

$$y_k(\lambda, x) = \int_{\mathbb{R}} G(x, \xi, \lambda) \widehat{v}_k(\xi) d\xi.$$

There is  $N_0 \in \mathbb{N}$  with  $J_N \supset \text{supp}(\mu_j)$  for all  $N \geq N_0$  and  $j = 1, \dots, m$ . Using (4.3) and (4.18) we find for  $N \geq N_0$

$$\begin{aligned} & |\langle \widehat{w}_j, y_k(\lambda) \rangle - \langle \widehat{w}_j|_{J_N}, y_k^N(\lambda) \rangle| \\ & \leq \left| \langle \widehat{w}_j, y_k(\lambda) \rangle - \langle \widehat{w}_j|_{J_N}, p_N y_k(\lambda) \rangle \right| + \left| \langle \widehat{w}_j|_{J_N}, p_N y_k(\lambda) - y_k^N(\lambda) \rangle \right| \\ & \leq c \|p_N y_k(\lambda) - y_k^N(\lambda)\|_{\mathcal{H}_N}. \end{aligned} \quad (4.23)$$

Since  $\widehat{v}_k$  has compact support and is bounded there is a constant  $c$  such that  $|\widehat{v}_k(\xi)| \leq ce^{-2\alpha|\xi|}$  for all  $\xi \in \mathbb{R}$  and  $k = 1, \dots, \ell$ . This is used to bound the right hand side of (4.12):

$$\begin{aligned} |R_-(\lambda)y_k(\lambda, x_-^N)| & \leq c \|R_-(\lambda)\| \int_{-\infty}^{x_-^N} e^{-\alpha|x_-^N - \xi|} |\widehat{v}_k(\xi)| d\xi \\ & \quad + c \|R_-(\lambda)(P(\lambda, x_-^N) - I)\| \int_{x_-^N}^{\infty} e^{-\alpha(\xi - x_-^N)} |\widehat{v}_k(\xi)| d\xi \\ & \leq c \int_{-\infty}^{x_-^N} e^{-\alpha|x_-^N - \xi|} e^{-2\alpha|\xi|} d\xi \\ & \quad + c \|R_-(\lambda)(P(\lambda, x_-^N) - I)\| e^{\alpha x_-^N} \int_{x_-^N}^{\infty} e^{-\alpha\xi} e^{-2\alpha|\xi|} d\xi \\ & \leq ce^{2\alpha x_-^N} + c \|R_-(\lambda)(P(\lambda, x_-^N) - I)\| e^{\alpha x_-^N}. \end{aligned} \quad (4.24)$$

A similar estimate holds for  $|R_+(\lambda)y_k(\lambda, x_+^N)|$ . Since the estimates are uniform in  $\lambda \in \Gamma$ , we obtain

$$\sup_{\lambda \in \Gamma} |R_-(\lambda)y_k(\lambda, x_-^N) + R_+(\lambda)y_k(\lambda, x_+^N)| \leq ce^{-\alpha \min\{-x_-^N, x_+^N\}}, \quad (4.25)$$

by Theorem 4.4 this proves (4.19).

If Hypothesis 4.5 holds, the projections  $P(\lambda, x)$  of the exponential dichotomy on the whole real line can be chosen to satisfy (4.14), again see [BL, Thm. A.5]. For projection boundary conditions we then find from (4.14) and (4.17)

$$\|R_-(\lambda)(P(\lambda, x_-^N) - I)\| = \|R_-(\lambda)(P(\lambda, x_-^N) - P_-(\lambda))\| \leq ce^{\alpha x_-^N}.$$

Summarizing, we can bound the right hand side of (4.12) as follows

$$\sup_{\lambda \in \Gamma} |R_-(\lambda)y_k(\lambda, x_-^N) + R_+(\lambda)y_k(\lambda, x_+^N)| \leq ce^{-2\alpha \min\{-x_-^N, x_+^N\}},$$

which gives the desired improved order of convergence.  $\blacksquare$

*Remark 4.9.* It is not difficult to weaken the assumption of compact support for  $\hat{v}_k$  and  $\hat{w}_j$ . For example, the proof in (4.24) shows that it is sufficient to have  $|\hat{v}_k(\xi)| \leq ce^{-2\alpha|\xi|}$  for  $\xi \in \mathbb{R}$  and  $k = 1, \dots, \ell$ .  $\diamond$

**4.2. Estimates of eigenvalues.** We analyze the error of the numerical algorithm from Section 2.2 when the matrices  $D_0, D_1$  are replaced by their approximations  $D_0^N, D_1^N$  satisfying the estimates (4.22).

As before, let  $D_0$  be of rank  $\varkappa$  and let  $D_0 = V_0 \Sigma_0 W_0^*$ , be the short form of its singular value decomposition (SVD), cf. (2.16). In the following we consider a small perturbation  $\tilde{D}_0 \in \mathbb{C}^{m, \ell}$  of  $D_0$  with the full SVD

$$\tilde{D}_0 = \begin{pmatrix} \tilde{V}_0 & \tilde{V}_1 \end{pmatrix} \begin{pmatrix} \tilde{\Sigma}_0 & 0_{\varkappa, \ell - \varkappa} \\ 0_{m - \varkappa, \varkappa} & \tilde{\Sigma}_1 \end{pmatrix} \begin{pmatrix} \tilde{W}_0^* \\ \tilde{W}_1^* \end{pmatrix}, \quad (4.26)$$

where  $\tilde{V}_0 \in \mathbb{C}^{m, \varkappa}, \tilde{V}_1 \in \mathbb{C}^{m, m - \varkappa}, \tilde{W}_0 \in \mathbb{C}^{\ell, \varkappa}, \tilde{W}_1 \in \mathbb{C}^{\ell, \ell - \varkappa}$ , and  $\tilde{\Sigma}_0 \in \mathbb{C}^{\varkappa, \varkappa}$  contains the  $\varkappa$  largest singular values of  $\tilde{D}_0$ . Instead of computing the eigenvalues of

$$D = V_0^* D_1 W_0 \Sigma_0^{-1} \quad (4.27)$$

in (2.18), we use (4.26) and compute the eigenvalues of

$$\tilde{D} = \tilde{V}_0^* \tilde{D}_1 \tilde{W}_0 \tilde{\Sigma}_0^{-1}, \quad (4.28)$$

where  $\tilde{D}_1$  is a small perturbation of  $D_1$ . The following lemma shows that the eigenvalues of  $\tilde{D}$  approximate those of  $D$  with the order of the original perturbations. In order to apply the perturbation theory from [St] we use the Frobenius norm  $\|D\|_F^2 = \text{tr}(D^* D)$ , the spectral norm  $\|D\|_2$  and the Hausdorff distance

$$\text{dist}_H(M_1, M_2) = \max\left(\sup_{z \in M_1} \inf_{y \in M_2} |z - y|, \sup_{z \in M_2} \inf_{y \in M_1} |z - y|\right), \quad M_1, M_2 \subset \mathbb{C}.$$

**Lemma 4.10.** *Let  $D_0, D_1 \in \mathbb{C}^{m, \ell}$  be given such that  $\text{rank } D_0 = \varkappa$  and such that the matrix  $D$  from (4.27) has only simple eigenvalues. Then there exist  $\varepsilon_0 > 0$  and  $C > 0$  such that the spectra of  $D$  and  $\tilde{D}$  from (4.28) satisfy*

$$\text{dist}_H\left(\sigma(D), \sigma(\tilde{D})\right) \leq C(\|D_0 - \tilde{D}_0\|_F + \|D_1 - \tilde{D}_1\|_F), \quad (4.29)$$

*provided  $\|D_0 - \tilde{D}_0\|_F + \|D_1 - \tilde{D}_1\|_F \leq \varepsilon_0$ .*

*Proof.* In the following  $C$  denotes a generic constant depending on  $\varepsilon_0$  but not on  $\tilde{D}_0$ ,  $\tilde{D}_1$  and  $\varepsilon := \|D_0 - \tilde{D}_0\|_F + \|D_1 - \tilde{D}_1\|_F \leq \varepsilon_0$ .

Let us extend  $Y_0 = V_0$  and  $X_0 = W_0$  to unitary matrices  $Y = (Y_0 \ Y_1) \in \mathbb{C}^{m,m}$  and  $X = (X_0 \ X_1) \in \mathbb{C}^{\ell,\ell}$  and introduce  $E \in \mathbb{C}^{m,\ell}$  such that

$$Y^* D_0 X = \begin{pmatrix} \Sigma_0 & 0_{\varkappa,\ell-\varkappa} \\ 0_{m-\varkappa,\varkappa} & 0_{m-\varkappa,\ell-\varkappa} \end{pmatrix}, \quad E = \begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix} := Y^* (\tilde{D}_0 - D_0) X$$

with conformal partitioning. We require  $4\varepsilon_0 < \sigma_{\min} = \min_{j=1,\dots,\varkappa} \sigma_j$  and obtain

$$\begin{aligned} \gamma &:= (\|E_{12}\|_F^2 + \|E_{21}\|_F^2)^{1/2} \leq \|E\|_F \leq \varepsilon \leq \varepsilon_0, \\ 2\varepsilon &\leq 2\varepsilon_0 < \frac{\sigma_{\min}}{2} \leq \sigma_{\min} - \sqrt{2}\varepsilon_0 \leq \sigma_{\min} - \|E_{11}\|_2 - \|E_{22}\|_2 =: \delta. \end{aligned}$$

Therefore [St, Thm. 6.4] applies and yields  $Q \in \mathbb{C}^{m-\varkappa,\varkappa}$ ,  $P \in \mathbb{C}^{\ell-\varkappa,\varkappa}$  with

$$(\|Q\|_F^2 + \|P\|_F^2)^{1/2} \leq 2\frac{\gamma}{\delta} \leq \frac{4}{\sigma_{\min}}\varepsilon, \quad (4.30)$$

so that the unitary matrices

$$\begin{aligned} \tilde{Y} &= (\tilde{Y}_0 \ \tilde{Y}_1) = Y \begin{pmatrix} I & -Q^* \\ Q & I \end{pmatrix} \begin{pmatrix} (I + Q^*Q)^{-1/2} & 0_{\varkappa,m-\varkappa} \\ 0_{m-\varkappa,\varkappa} & (I + QQ^*)^{-1/2} \end{pmatrix}, \\ \tilde{X} &= (\tilde{X}_0 \ \tilde{X}_1) = X \begin{pmatrix} I & -P^* \\ P & I \end{pmatrix} \begin{pmatrix} (I + P^*P)^{-1/2} & 0_{\varkappa,\ell-\varkappa} \\ 0_{\ell-\varkappa,\varkappa} & (I + PP^*)^{-1/2} \end{pmatrix} \end{aligned} \quad (4.31)$$

transform  $\tilde{D}_0$  into block diagonal form

$$\tilde{Y}^* \tilde{D}_0 \tilde{X} = \begin{pmatrix} A_{11} & 0_{\varkappa,\ell-\varkappa} \\ 0_{m-\varkappa,\varkappa} & A_{22} \end{pmatrix} \in \mathbb{C}^{m,\ell}.$$

The matrices  $A_{11}$  and  $A_{22}$  can be written as

$$\begin{aligned} A_{11} &= (I + Q^*Q)^{1/2} (\Sigma_0 + E_{11} + E_{12}P) (I + P^*P)^{-1/2}, \\ A_{22} &= (I + QQ^*)^{1/2} (E_{22} - E_{21}P^*) (I + PP^*)^{-1/2}. \end{aligned} \quad (4.32)$$

From the estimate (4.30) we find

$$\|(I + Q^*Q)^{\pm 1/2} - I\|_2 \leq C\|Q\|_2^2 \leq C\varepsilon^2$$

and a similar estimate holds for  $(I + QQ^*)^{\pm 1/2}$ . Using (4.31), (4.32) this leads to

$$\|\tilde{Y}_0 - V_0\|_F \leq C\varepsilon, \quad \|\tilde{X}_0 - W_0\|_F \leq C\varepsilon, \quad \|\Sigma_0 - A_{11}\|_2 \leq C\varepsilon, \quad \|A_{22}\|_2 \leq C\varepsilon. \quad (4.33)$$

By [GK, II Cor. 2.3] and the last two inequalities in (4.33) the sets of singular values  $\text{sing } A_{11}$  of  $A_{11}$  and  $\text{sing } A_{22}$  of  $A_{22}$  satisfy

$$\min(\text{sing } A_{11}) \geq \sigma_{\min} - C\varepsilon, \quad \max(\text{sing } A_{22}) \leq C\varepsilon.$$

Decreasing  $\varepsilon_0$  further we find that  $\text{sing } A_{11}$  and  $\text{sing } A_{22}$  are disjoint and hence  $\text{sing } A_{11}$  contains the  $\varkappa$  largest singular values of  $\tilde{D}_0$ .

Therefore,  $\mathcal{R}(\tilde{X}_0)$  is the invariant subspace of  $\tilde{D}_0^* \tilde{D}_0$  corresponding to the  $\varkappa$  largest eigenvalues of  $\tilde{D}_0^* \tilde{D}_0$  and coincides with  $\mathcal{R}(\tilde{W}_0)$ . Similarly,  $\mathcal{R}(\tilde{Y}_0) = \mathcal{R}(\tilde{V}_0)$ . Then the matrices  $T_0 = \tilde{X}_0^* \tilde{W}_0 \in \mathbb{C}^{\varkappa,\varkappa}$  and  $S_0 = \tilde{Y}_0^* \tilde{V}_0 \in \mathbb{C}^{\varkappa,\varkappa}$  are unitary and satisfy  $\tilde{X}_0 T_0 = \tilde{W}_0$  and  $\tilde{Y}_0 S_0 = \tilde{V}_0$ . Moreover,  $\tilde{\Sigma}_0 = S_0^* \tilde{Y}_0^* \tilde{D}_0 \tilde{X}_0 T_0 = S_0^* A_{11} T_0$ , so that the matrices

$$\tilde{D} = \tilde{V}_0^* \tilde{D}_1 \tilde{W}_0 \tilde{\Sigma}_0^{-1} = S_0^* \tilde{Y}_0^* \tilde{D}_1 \tilde{X}_0 T_0 T_0^* A_{11}^{-1} S_0 = S_0^* \tilde{Y}_0^* \tilde{D}_1 \tilde{X}_0 A_{11}^{-1} S_0$$

and

$$\widehat{D} = \widetilde{Y}_0^* \widetilde{D}_1 \widetilde{X}_0 A_{11}^{-1}$$

are similar and have the same spectrum. Now estimates (4.33) imply

$$\|\widehat{D} - D\|_F \leq C\varepsilon.$$

Since simple eigenvalues depend analytically on the matrix, we finally obtain for some  $C > 0$

$$\text{dist}_H(\sigma(D), \sigma(\widehat{D})) = \text{dist}_H(\sigma(D), \sigma(\widehat{D})) \leq C\varepsilon. \quad (4.34)$$

This finishes the proof.  $\blacksquare$

Combining this result with Corollary 4.8 shows that using boundary value problems for the computation of the point spectrum is a robust method and leads to exponential convergence with respect to the length of intervals.

**Theorem 4.11.** *Let the assumptions of Theorem 4.7 hold and let all eigenvalues of  $F$  inside the contour  $\Gamma$  be simple. Then there is  $C > 0$  such that the following holds for all intervals  $[x_-^N, x_+^N]$  with  $\min\{-x_-^N, x_+^N\}$  sufficiently large. The set  $\sigma^N$  of the eigenvalues of the approximate pencil  $F_N$  from (4.1), computed by the method from Section 2.2 using the approximations  $D_0^N, D_1^N$  from (4.21) instead of  $D_0, D_1$ , satisfies the estimate*

$$\text{dist}_H(\sigma^N, \sigma(D)) \leq Ce^{-\alpha \min\{-x_-^N, x_+^N\}}. \quad (4.35)$$

*In case of projection boundary conditions the constant  $\alpha$  improves to  $2\alpha$ .*

*Remark 4.12.* We note that the simplicity of eigenvalues was only used in the very last step (4.34) of the proof of Lemma 4.10. Similar to Remark 2.9 convergence of spectra as  $\varepsilon \rightarrow 0$  still follows in the general case from the perturbation theory in [K]. But now the rate is  $\varepsilon^{\frac{1}{\mu}}$  in (4.34) where  $\mu$  is the maximal algebraic multiplicity of eigenvalues inside  $\Gamma$ . Correspondingly, the rate  $\alpha$  in (4.35) deteriorates to  $\frac{\alpha}{\mu}$ .

## 5. THE SCHRÖDINGER OPERATOR ON THE LINE

We consider the eigenvalue problem for the one dimensional Schrödinger operator,  $H$ ,

$$(H - \lambda)u = 0, \quad H = -d^2/dx^2 + V(x), \quad x \in \mathbb{R}. \quad (5.1)$$

Here, the real valued potential  $V$  satisfies  $V \in L^1(\mathbb{R})$ , the domain of  $H$  is given by

$$\text{dom}(H) = \{u \in L^2(\mathbb{R}) : u, u' \in AC_{\text{loc}}(\mathbb{R}), -u'' + Vu \in L^2(\mathbb{R})\},$$

and we assume that  $\lambda \in \Omega = \mathbb{C} \setminus [0, \infty)$ . Since  $V \in L^1(\mathbb{R})$ , the essential spectrum of  $H$  is equal to  $[0, \infty)$ , and the discrete spectrum consists of no more than finitely many negative simple eigenvalues  $0 > \lambda_1 > \lambda_2 > \dots > \lambda_\infty$ , see, e.g., [RSIV, Sec.XIII.3], [CS, Sec.XVII.1.3]. The eigenvalue problem (5.1) can be written as the first order differential equation

$$y' = A(\lambda, x)y, \quad A(\lambda, x) = A(\lambda, \infty) + B(x), \quad x \in \mathbb{R}; \quad (5.2)$$

here and below we denote

$$A(\lambda, \infty) = \begin{pmatrix} 0 & 1 \\ -\lambda & 0 \end{pmatrix}, \quad B(x) = \begin{pmatrix} 0 & 0 \\ V(x) & 0 \end{pmatrix}, \quad y(x) = \begin{pmatrix} u(x) \\ u'(x) \end{pmatrix}. \quad (5.3)$$

In Subsection 5.1 we consider the linear operator pencil of second order differential operators  $F^{(\text{II})}(\lambda) = H - \lambda I$ , see (5.1), acting from the space  $\mathcal{H} = \text{dom}(H)$  equipped

with the graph norm into the space  $\mathcal{K} = L^2(\mathbb{R}, \mathbb{C}^2)$ . In Subsection 5.2 we consider the nonlinear operator pencil of first order differential operators  $F^{(I)}(\lambda) = -\partial_x + A(\lambda, \cdot)$ , see (5.2), (5.3), acting from the space  $\mathcal{H}$  as defined in (3.37) into the space  $\mathcal{K} = L^2(\mathbb{R}, \mathbb{C}^2)$ . Our objective is to illustrate the construction of the matrix  $E$  from (2.4), (3.51), (3.53) and also its computation via some approximation arguments related to the boundary value problems on finite intervals, that is, to the equation

$$-u''(x) + V(x)u(x) - \lambda u(x) = 0, \quad x \in [x_-^N, x_+^N], \quad (5.4)$$

equipped with appropriate boundary conditions at the endpoints  $x_-^N, x_+^N$  satisfying  $x_-^N \rightarrow -\infty$  and  $x_+^N \rightarrow +\infty$  as  $N \rightarrow \infty$ , and to the boundary value problems for the first order differential equation (5.2). In the current section we do not assume exponential decay of the perturbation. Although we offer some explicit formulas for the matrix  $E$  and its approximation  $E^N$  in terms of certain solutions of the differential equations (5.1), (5.2), (5.3), we emphasize that they are mainly of theoretical value as our general approach in practical applications is *not* to use these formulas but instead to construct  $E^N$  by solving boundary value problems on finite intervals numerically.

**5.1. Second order differential operators.** We consider the linear operator pencil  $F^{(II)}(\lambda) = H - \lambda I$  with  $H$  as in (5.1). Our main tool will be the Jost solutions  $u_{\pm}(\lambda, x)$ ,  $x \in \mathbb{R}$ ,  $\lambda \in \Omega = \mathbb{C} \setminus [0, \infty)$ , of the second order Schrödinger differential equation (5.1) which are uniquely determined as the solutions of the Volterra integral equations

$$u_{\pm}(\lambda, x) = e^{\pm i\lambda^{1/2}x} - \int_0^{\pm\infty} \lambda^{-1/2} \sin(\lambda^{1/2}(x - \xi)) V(\xi) u_{\pm}(\lambda, \xi) d\xi, \quad x \in \mathbb{R}. \quad (5.5)$$

Here and everywhere below we choose the branch of the square root such that  $\text{Im}(\lambda^{1/2}) > 0$  for  $\lambda \in \Omega$ , in particular,  $e^{i\lambda^{1/2}x} \rightarrow 0$  as  $x \rightarrow +\infty$ . It is well known that the Jost solutions satisfy the asymptotic boundary conditions

$$\lim_{x \rightarrow \pm\infty} e^{\mp i\lambda^{1/2}x} u_{\pm}(\lambda, x) = 1, \quad (5.6)$$

they are holomorphic functions of  $\lambda \in \Omega$ , for  $\lambda < 0$  they are real valued and positive for  $\pm x$  sufficiently large, see, e.g., [CS, Chap.XVII]. The Wronskian

$$\mathcal{W}(\lambda) = \mathcal{W}(u_-, u_+) = u_-(\lambda, x) u'_+(\lambda, x) - u'_-(\lambda, x) u_+(\lambda, x), \quad x \in \mathbb{R}, \lambda \in \Omega, \quad (5.7)$$

of the Jost solutions is equal to zero precisely at the points  $\lambda_n \in \Omega$ , the isolated eigenvalues of the Schrödinger operator  $H$ , where the exponentially decaying at  $+\infty$  solution  $u_+(\lambda_n, \cdot)$  is proportional to the exponentially decaying at  $-\infty$  solution  $u_-(\lambda_n, \cdot)$  with a nonzero constant  $c_n$ , that is, when

$$u_+(\lambda_n, x) = c_n u_-(\lambda_n, x), \quad x \in \mathbb{R}, c_n \in \mathbb{C} \setminus \{0\}, n = 1, \dots, \varkappa. \quad (5.8)$$

We refer to [W87] for the general theory of Sturm-Liouville differential operators (see also [W05] for a brief but exceptionally readable account). In particular, due to  $V \in L^1(\mathbb{R})$  the Schrödinger operator  $H$  is in the limit point case at  $\pm\infty$ , the Jost solutions  $u_{\pm}$  are  $L^2$ -solutions at  $\pm\infty$ , and thus the resolvent operator  $(F^{(II)}(\lambda))^{-1} = (H - \lambda I)^{-1}$  for  $\lambda \in \Omega \setminus \{\lambda_1, \dots, \lambda_{\varkappa}\}$  is the integral operator with the kernel

$$R(\lambda, x, \xi) = \frac{1}{\mathcal{W}(u_+, u_-)} \begin{cases} u_+(\lambda, x) u_-(\lambda, \xi), & -\infty < \xi \leq x < +\infty, \\ u_-(\lambda, x) u_+(\lambda, \xi), & -\infty < x < \xi < +\infty. \end{cases} \quad (5.9)$$

Therefore, if  $\widehat{w}_j, \widehat{v}_k \in L^2(\mathbb{R})$  are chosen as indicated in Section 2, that is, such that

$$\text{rank} \left( \langle \widehat{w}_j(\cdot), u_{\pm}(\lambda_n, \cdot) \rangle_{\mathbb{R}} \right)_{j,n=1}^{m,\varkappa} \geq \varkappa, \text{rank} \left( \langle u_{\pm}(\lambda_n, \cdot), \widehat{v}_k(\cdot) \rangle_{\mathbb{R}} \right)_{n,k=1}^{\varkappa,\ell} \geq \varkappa,$$

then the matrix  $E(\lambda) = \left( \langle \widehat{w}_j(\cdot), ((F^{(\text{II})}(\lambda))^{-1} \widehat{v}_k)(\cdot) \rangle_{\mathbb{R}} \right)_{j,k=1}^{m,\ell}$  from (2.4) is given by

$$\begin{aligned} E_{jk}(\lambda) &= \frac{1}{\mathcal{W}(u_+, u_-)} \int_{-\infty}^{\infty} \widehat{w}_j(x) u_+(\lambda, x) \int_{-\infty}^x u_-(\lambda, \xi) \widehat{v}_k(\xi) d\xi dx \\ &\quad + \frac{1}{\mathcal{W}(u_+, u_-)} \int_{-\infty}^{\infty} \widehat{w}_j(x) u_-(\lambda, x) \int_x^{\infty} u_+(\lambda, \xi) \widehat{v}_k(\xi) d\xi dx. \end{aligned} \quad (5.10)$$

Integrating  $E(\lambda)$  from (5.10) over the contour  $\Gamma$  from Section 2.2 we thus obtain  $\varkappa = \text{rank } D_0$  and formulas (2.18), (2.19) for the eigenvalues of  $H$ .

We now equip equation (5.4) with self-adjoint boundary conditions

$$u(x_{\pm}^N) \cos \omega_{\pm} - u'(x_{\pm}^N) \sin \omega_{\pm} = 0, \quad \text{with some } \omega_{\pm} \in [0, \pi), \quad (5.11)$$

and define the operator  $H^N$  in  $L^2([x_-^N, x_+^N])$  by  $H^N = -d^2/dx^2 + V(x)$  with

$$\begin{aligned} \text{dom}(H^N) &= \{u \in L^2([x_-^N, x_+^N]) : u, u' \in AC_{\text{loc}}([x_-^N, x_+^N]), -u'' + Vu \in L^2([x_-^N, x_+^N]) \\ &\quad \text{and both boundary conditions (5.11) hold}\}, \end{aligned}$$

cf. [W05, Sec.7]. Let  $\tilde{u}_{\pm}(\lambda, x)$  denote the  $N$ -dependent solutions of the Schrödinger equation (5.1) each of them satisfying one of the respective initial conditions

$$\tilde{u}_{\pm}(\lambda, x_{\pm}^N) = e^{i\lambda^{1/2} x_{\pm}^N} \sin \omega_{\pm}, \quad \tilde{u}'_{\pm}(\lambda, x_{\pm}^N) = e^{i\lambda^{1/2} x_{\pm}^N} \cos \omega_{\pm}. \quad (5.12)$$

Since  $\tilde{u}_+(\lambda, x)$ , respectively,  $\tilde{u}_-(\lambda, x)$  satisfies the boundary condition (5.11) at  $x_+^N$ , respectively,  $x_-^N$  we conclude (see, e.g., [W05, p.84]) that the resolvent operator  $(F^{(\text{II},N)}(\lambda))^{-1} = (H^N - \lambda I)^{-1}$  is the integral operator with the kernel

$$R^N(\lambda, x, \xi) = \frac{1}{\mathcal{W}(\tilde{u}_+, \tilde{u}_-)} \begin{cases} \tilde{u}_+(\lambda, x) \tilde{u}_-(\lambda, \xi), & x_-^N \leq \xi \leq x \leq x_+^N, \\ \tilde{u}_-(\lambda, x) \tilde{u}_+(\lambda, \xi), & x_-^N \leq x < \xi \leq x_+^N. \end{cases} \quad (5.13)$$

In the following we consider the restriction  $p_N u = u|_{[x_-^N, x_+^N]}$  (cf. (4.3)) as an operator from  $L^2(\mathbb{R})$  into  $L^2([x_-^N, x_+^N])$ . Then it is known from [W05, Thm. 7.1] that  $(F^{(\text{II},N)}(\lambda))^{-1} p_N$  converges strongly to  $(F^{(\text{II})}(\lambda))^{-1}$  in  $L^2(\mathbb{R})$ . This is called generalized strong resolvent convergence of  $H^N$  to  $H$  in [W05] (here, one imbeds  $L^2([x_-^N, x_+^N])$  into  $L^2(\mathbb{R})$  by setting functions equal to zero in  $\mathbb{R} \setminus [x_-^N, x_+^N]$ ). It follows that the matrix  $E(\lambda)$  from (2.4) for the operator pencil  $F^{(\text{II})}(\lambda)$  can be written as the limit of the matrices  $E^N(\lambda)$  defined via the approximative operator pencils  $F^{(\text{II},N)}(\lambda)$ .

**Proposition 5.1.** *Assume  $V \in L^1(\mathbb{R})$  and let  $E(\lambda)$  be defined as in (5.10). Then*

$$E(\lambda) = \lim_{N \rightarrow \infty} E^N(\lambda) \quad \text{where} \quad E^N(\lambda) = \left( \int_{x_-^N}^{x_+^N} \widehat{w}_j^N(x) ((F^{(\text{II},N)}(\lambda))^{-1} \widehat{v}_k^N)(x) dx \right)_{j,k=1}^{m,\ell},$$



and we denote  $\widehat{w}_j^N = p_N \widehat{w}_j$ ,  $\widehat{v}_k^N = p_N \widehat{v}_k$ . Similarly to (5.10), using (5.13) the matrix  $E^N(\lambda)$  can be computed by the formula

$$\begin{aligned} E_{jk}^N(\lambda) &= \frac{1}{\mathcal{W}(\widetilde{u}_+, \widetilde{u}_-)} \int_{x_-^N}^{x_+^N} \widehat{w}_j^N(x) \widetilde{u}_+(\lambda, x) \int_{x_-^N}^x \widetilde{u}_-(\lambda, \xi) \widehat{v}_k^N(\xi) d\xi dx \\ &+ \frac{1}{\mathcal{W}(\widetilde{u}_+, \widetilde{u}_-)} \int_{x_-^N}^{x_+^N} \widehat{w}_j^N(x) \widetilde{u}_-(\lambda, x) \int_x^{x_+^N} \widetilde{u}_+(\lambda, \xi) \widehat{v}_k^N(\xi) d\xi dx. \end{aligned} \quad (5.14)$$

We note in passing that  $\lim_{N \rightarrow \infty} \mathcal{W}(\widetilde{u}_+, \widetilde{u}_-) = C(\lambda) \mathcal{W}(u_+, u_-)$  where the factor  $C(\lambda)$  can be explicitly computed and is equal to zero precisely at the eigenvalues of the operators  $H_+^0$  and  $H_-^0$  defined as  $H_\pm^0 = -d^2/dx^2$  on  $L^2((-\infty, x_+^N])$  and  $L^2([x_-^N, \infty))$  with the domain determined by the respective boundary condition in (5.11), see [LS, Thm.3.3].

**5.2. First order differential operators.** With  $A(\lambda, x)$  as in (5.2), (5.3) we consider the operator pencil  $F^{(1)}(\lambda)y = -y' + A(\lambda, x)y$  and follow step-by-step the constructions in Subsections 3.1 and 3.2 culminating in formulas (3.51) and (3.53).

First, we need to choose the projections on the subspaces of the initial values of the solutions of (5.2) exponentially decaying at  $+\infty$  and  $-\infty$  and construct their representation (3.16) and the respective Evans function (3.18). The matrix  $A(\lambda, \infty)$  for  $\lambda \in \Omega$  has no pure imaginary eigenvalues and the differential equation  $y' = A(\lambda, \infty)y$  has the exponential dichotomy on  $\mathbb{R}$  with the dichotomy projection being the spectral projection  $P(\lambda, \infty) := P_+(\lambda) = P_-(\lambda)$  for  $A(\lambda, \infty)$  corresponding to the eigenvalue  $i\lambda^{1/2}$ . We recall that if  $\lambda \in \Omega$  then the eigenvalues  $\pm i\lambda^{1/2}$  of  $A(\lambda, \infty)$  satisfy  $\operatorname{Re}(i\lambda^{1/2}) < 0 < \operatorname{Re}(-i\lambda^{1/2})$ .

Since the perturbation  $B$  in (5.2) satisfies  $\|B(\cdot)\| \in L^1(\mathbb{R})$  by the general theory in e.g. [BL, Co, GLM] there exist dichotomy projections

$$P_\pm(\lambda, x) = S(x, 0, \lambda) P_\pm(\lambda, 0) S(0, x, \lambda) \quad (5.15)$$

on  $\mathbb{R}_\pm$  for the perturbed equation  $y' = A(\lambda, x)y$  such that the dichotomy subspaces  $\mathcal{R}(P_+(\lambda, x)) = \operatorname{span}\{y_+(\lambda, x)\}$  and  $\mathcal{N}(P_-(\lambda, x)) = \operatorname{span}\{y_-(\lambda, x)\}$  are uniquely determined while their direct complements  $\mathcal{N}(P_+(\lambda, x))$  and  $\mathcal{R}(P_-(\lambda, x))$  are arbitrary. Here and below we use notation

$$y_\pm(\lambda, x) = \begin{pmatrix} u_\pm(\lambda, x) \\ u'_\pm(\lambda, x) \end{pmatrix}, \quad x \in \mathbb{R}, \quad \lambda \in \Omega, \quad (5.16)$$

for the  $(2 \times 1)$  vector solutions of (5.2), (5.3) which correspond to the Jost solutions  $u_\pm(\lambda, x)$  of (5.1) defined via the Volterra equations (5.5). Also, given a vector  $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathbb{C}^2$  we denote  $v^\dagger = \begin{pmatrix} -v_2 \\ v_1 \end{pmatrix}$  and  $v^\perp = (v^\dagger)^\top = (-v_2 \ v_1)$  so that for any two vectors  $v, w \in \mathbb{C}^2$  we have  $\det(v|w) = v^\perp w = -v^\top w^\dagger$ ; thus, if  $v, w \in \mathbb{C}^2$  are linearly independent then the projection on  $\operatorname{span}\{v\}$  parallel to  $\operatorname{span}\{w\}$  is the matrix  $(w^\perp v)^{-1} v w^\perp$ .

To make a choice of the dichotomy projections  $P_\pm(\lambda, x)$  one is tempted to let

$$\begin{aligned} P_+(\lambda, 0) &= (\mathcal{W}(u_-, u_+))^{-1} y_+(\lambda, 0) y_-(\lambda, 0)^\perp, \\ I - P_-(\lambda, 0) &= (\mathcal{W}(u_+, u_-))^{-1} y_-(\lambda, 0) y_+(\lambda, 0)^\perp \end{aligned} \quad (5.17)$$

and then use (5.15) to define  $P_\pm(\lambda, x)$  for  $x \in \mathbb{R}_\pm$ . This choice, however, is not satisfactory as these projections are meromorphic in  $\Omega$  with the poles precisely at the

eigenvalues  $\lambda_n$  while the constructions in Subsection 3 require holomorphy. Another choice is to normalize  $u_{\pm}$  by letting  $\hat{u}_{\pm}(\lambda, x) = (u_{\pm}^2(\lambda, 0) + u'_{\pm}^2(\lambda, 0))^{-1/2} u_{\pm}(\lambda, x)$  and then replace  $u_{\pm}$  and  $y_{\pm}$  in (5.17) by  $\hat{u}_{\pm}$  and  $\hat{y}_{\pm}$ . This choice yields holomorphy of  $P_{\pm}(\lambda, x)$  in  $\Omega \setminus \Lambda$ , where  $\Lambda = \{\lambda : (u_+^2(\lambda, 0) + u'_+{}^2(\lambda, 0))(u_-^2(\lambda, 0) + u'_-{}^2(\lambda, 0)) = 0\}$ , and admits the normalization of the type (3.4), (3.12) leading to the construction of the normalized Evans function  $\mathcal{E}_0$ . However, the great disadvantage of this choice of  $P_{\pm}(\lambda, x)$  is that these dichotomy projections are not asymptotic as  $x \rightarrow \pm\infty$  to the spectral projections of  $A(\lambda, \infty)$ .

We now construct the holomorphic dichotomy projections that are asymptotic to the spectral projections at infinity. Our main tool will be the solutions  $y_+^s(\lambda, \cdot)$ ,  $y_+^u(\lambda, \cdot)$  on  $\mathbb{R}_+$  and  $y_-^s(\lambda, \cdot)$ ,  $y_-^u(\lambda, \cdot)$  on  $\mathbb{R}_-$  of the differential equation (5.2) satisfying the asymptotic boundary conditions

$$\lim_{x \rightarrow +\infty} e^{-i\lambda^{1/2}x} y_+^s(\lambda, x) = \mathbf{v}, \quad \lim_{x \rightarrow +\infty} e^{i\lambda^{1/2}x} y_+^u(\lambda, x) = \mathbf{w}, \quad (5.18)$$

$$\lim_{x \rightarrow -\infty} e^{i\lambda^{1/2}x} y_-^s(\lambda, x) = \mathbf{w}, \quad \lim_{x \rightarrow -\infty} e^{-i\lambda^{1/2}x} y_-^u(\lambda, x) = \mathbf{v}, \quad (5.19)$$

where  $\mathbf{v} = \begin{pmatrix} 1 \\ i\lambda^{1/2} \end{pmatrix}$ ,  $\mathbf{w} = \begin{pmatrix} 1 \\ -i\lambda^{1/2} \end{pmatrix}$  are the eigenvectors of  $A(\lambda, \infty) = \begin{pmatrix} 0 & 1 \\ -\lambda & 0 \end{pmatrix}$  such that  $A(\lambda, \infty)\mathbf{v} = i\lambda^{1/2}\mathbf{v}$ ,  $A(\lambda, \infty)\mathbf{w} = -i\lambda^{1/2}\mathbf{w}$ . The spectral projections of the matrix  $A(\lambda, \infty)$  are given by the formulas  $P(\lambda, \infty) = (2i\lambda^{1/2})^{-1}\mathbf{v}\mathbf{v}^\perp$  and  $I - P(\lambda, \infty) = -(2i\lambda^{1/2})^{-1}\mathbf{w}\mathbf{w}^\perp$ .

Since  $\|B(\cdot)\| \in L^1(\mathbb{R})$ , the existence of the solutions  $y_+^{s,u}(\lambda, \cdot)$  on  $\mathbb{R}_+$  satisfying (5.18) and the solutions  $y_-^{s,u}(\lambda, \cdot)$  on  $\mathbb{R}_-$  satisfying (5.19) is guaranteed by the celebrated Levinson theorem from asymptotic theory of differential equations (see, e.g., [CoL, Probl.III.29] or [E, Thms.1.3.1, 1.8.1] and also [GLM, Rem.7.11]). These solutions are obtained by solving certain inhomogeneous Fredholm type integral equations on semilines with holomorphic integral kernels and inhomogeneities. In fact, inspecting the proof of the theorem, cf. [E, Sec.1.4] or [GLM, Thm.8.3], we observe that the solutions  $y_{\pm}^{s,u}(\lambda, \cdot)$  are holomorphic in  $\lambda \in \Omega$  and limiting relations (5.18), (5.19) hold uniformly in  $\lambda$  on compact subsets of  $\Omega$ .

We will now discuss how the solutions  $y_+^{s,u}(\lambda, \cdot)$ ,  $y_-^{s,u}(\lambda, \cdot)$  satisfying (5.18), (5.19) are related to the solutions  $y_{\pm}(\lambda, \cdot)$  defined in (5.16) via the Jost solutions  $u_{\pm}(\lambda, \cdot)$  of (5.5). In fact, the solution  $y_+^s(\lambda, \cdot)$ , respectively,  $y_-^s(\lambda, \cdot)$  is uniquely determined by the first limiting relation in (5.18), respectively, (5.19) since due to (5.6) it is precisely the solution  $y_+(\lambda, \cdot)$ , respectively,  $y_-(\lambda, \cdot)$  defined in (5.16):

$$y_{\pm}^s(\lambda, x) = y_{\pm}(\lambda, x), \quad x \in \mathbb{R}_{\pm}, \quad \lambda \in \Omega. \quad (5.20)$$

The solution  $y_+^u(\lambda, \cdot)$ , respectively,  $y_-^u(\lambda, \cdot)$  is not unique and can be changed by adding a summand proportional to  $y_+(\lambda, \cdot)$ , respectively,  $y_-(\lambda, \cdot)$ . A convenient choice of the solutions  $y_{\pm}^u(\lambda, \cdot)$  is furnished by the formulas

$$\begin{aligned} y_+^u(\lambda, x) &= \frac{2i\lambda^{1/2}}{\mathcal{W}(u_-, u_+)} y_-(\lambda, x) - \sum_{n=1}^{\infty} \frac{2i\lambda^{1/2}\rho_n}{c_n(\lambda - \lambda_n)} y_+(\lambda, x), \quad x \in \mathbb{R}_+, \\ y_-^u(\lambda, x) &= \frac{2i\lambda^{1/2}}{\mathcal{W}(u_-, u_+)} y_+(\lambda, x) - \sum_{n=1}^{\infty} \frac{2i\lambda^{1/2}\rho_n c_n}{(\lambda - \lambda_n)} y_-(\lambda, x), \quad x \in \mathbb{R}_-, \end{aligned} \quad (5.21)$$

where  $y_{\pm}(\lambda, \cdot)$  are defined in (5.16),  $\lambda_n$ ,  $n = 1, \dots, \infty$ , are the eigenvalues of  $H$ , the constants  $c_n$  are taken from (5.8), and we denote by  $\rho_n$  the residue at  $\lambda_n$  of

the function  $1/\mathcal{W}(\lambda)$  for the Wronskian  $\mathcal{W}(\lambda)$  defined in (5.7). As we will see in a moment, the solutions  $y_{\pm}^s(\lambda, \cdot)$ ,  $y_{\pm}^u(\lambda, \cdot)$  defined in (5.20), (5.21) satisfy (5.18), (5.19). Using these solutions at  $x = 0$  we let

$$\begin{aligned} P_+(\lambda, 0) &= (2i\lambda^{1/2})^{-1} y_+^s(\lambda, 0) y_+^u(\lambda, 0)^{\perp}, \\ I - P_-(\lambda, 0) &= -(2i\lambda^{1/2})^{-1} y_-^s(\lambda, 0) y_-^u(\lambda, 0)^{\perp} \end{aligned} \quad (5.22)$$

and then use (5.15) to define  $P_{\pm}(\lambda, x)$  for  $x \in \mathbb{R}_{\pm}$ . By a direct calculation we also have  $P_-(\lambda, 0) = (2i\lambda^{1/2})^{-1} y_-^u(\lambda, 0) y_-^s(\lambda, 0)^{\perp}$ .

**Lemma 5.2.** *Assume that  $V \in L^1(\mathbb{R})$  and  $\lambda \in \Omega = \mathbb{C} \setminus [0, \infty)$ . Then the projections*

$$P_+(\lambda, x) = (2i\lambda^{1/2})^{-1} y_+^s(\lambda, x) y_+^u(\lambda, x)^{\perp}, \quad x \in \mathbb{R}_+, \quad (5.23)$$

$$P_-(\lambda, x) = (2i\lambda^{1/2})^{-1} y_-^u(\lambda, x) y_-^s(\lambda, x)^{\perp}, \quad x \in \mathbb{R}_-, \quad (5.24)$$

defined via formulas (5.15), (5.22) are holomorphic in  $\Omega$  and satisfy

$$\lim_{x \rightarrow \pm\infty} P_{\pm}(\lambda, x) = P(\lambda, \infty). \quad (5.25)$$

*Proof.* Formulas (5.18), (5.19) for the solutions  $y_{\pm}^s(\lambda, \cdot)$ ,  $y_{\pm}^u(\lambda, \cdot)$  defined in (5.20), (5.21) follow from (5.6) and the relations  $\lim_{x \rightarrow \pm\infty} e^{\mp i\lambda^{1/2}x} u'_{\pm}(\lambda, x) = \pm i\lambda^{1/2}$  and

$$\lim_{x \rightarrow \pm\infty} e^{\pm i\lambda^{1/2}x} u_{\mp}(\lambda, x) = \mathcal{W}(\lambda)/(2i\lambda^{1/2}), \quad \lim_{x \rightarrow \pm\infty} e^{\pm i\lambda^{1/2}x} u'_{\mp}(\lambda, x) = \mp \mathcal{W}(\lambda)/2,$$

(see, e.g., [LS, Lem.3.1]). Also, computing the residues of the RHS of (5.21) and using (5.8) we see that the solutions  $y_{\pm}^s(\lambda, \cdot)$ ,  $y_{\pm}^u(\lambda, \cdot)$  defined via (5.20), (5.21) are holomorphic in  $\lambda \in \Omega$ . In the remaining part of the proof we concentrate on the case of  $\mathbb{R}_+$  as the arguments for  $\mathbb{R}_-$  are similar. The Wronskian of the solutions  $y_+^s(\lambda, \cdot)$  and  $y_+^u(\lambda, \cdot)$  is  $x$ -independent and by (5.18), (5.19) we infer that

$$y_+^u(\lambda, x)^{\perp} y_+^s(\lambda, x) = (e^{i\lambda^{1/2}x} y_+^u(\lambda, x))^{\perp} (e^{-i\lambda^{1/2}x} y_+^s(\lambda, x)) \rightarrow \mathbf{w}^{\perp} \mathbf{v} = 2i\lambda^{1/2}$$

as  $x \rightarrow +\infty$ . Therefore,

$$y_+^u(\lambda, x)^{\perp} y_+^s(\lambda, x) = 2i\lambda^{1/2} \quad \text{for all } x \in \mathbb{R}_+ \quad (5.26)$$

and thus (5.23) is a holomorphic projection. But  $y_+^s(\lambda, x) = S(x, 0, \lambda) y_+^s(\lambda, 0)$  by the definition of  $S(x, 0, \lambda)$  and  $y_+^u(\lambda, x)^{\perp} = y_+^u(\lambda, 0)^{\perp} S(x, 0, \lambda)^{-1}$  since  $w(x) = y_+^u(\lambda, 0)^{\perp} S(x, 0, \lambda)^{-1}$  satisfies the adjoint equation  $w' = -wA(\lambda, x)^{\top}$  and therefore should be of the form  $w = y^{\perp}$  for a solution  $y$  of the equation  $y' = A(\lambda, x)y$ . Thus, (5.23) is in concert with (5.15) and (5.22). Using (5.18), (5.19) again we have

$$y_+^s(\lambda, x) y_+^u(\lambda, x)^{\perp} = (e^{-i\lambda^{1/2}x} y_+^s(\lambda, x)) (e^{i\lambda^{1/2}x} y_+^u(\lambda, x))^{\perp} \rightarrow \mathbf{v} \mathbf{w}^{\perp}$$

as  $x \rightarrow +\infty$  yielding (5.25). ■

We are ready to identify the ingredients in the representation (3.16) of the projections  $\Pi_U(\lambda) = P_+(\lambda, 0)$  and  $\Pi_V(\lambda) = I - P_-(\lambda, 0)$  on the holomorphic families of one dimensional subspaces  $U(\lambda) = \text{span}\{y_+(\lambda, 0)\}$  and  $V(\lambda) = \text{span}\{y_-(\lambda, 0)\}$ . Indeed, (5.20) and formulas (5.23), (5.24) show that (3.16) holds with

$$P(\lambda) = y_+^s(\lambda, 0), \quad \Phi(\lambda) = (2i\lambda^{1/2})^{-1} (y_+^u(\lambda, 0))^{\dagger}, \quad (5.27)$$

$$Q(\lambda) = -y_-^s(\lambda, 0), \quad \Psi(\lambda) = (2i\lambda^{1/2})^{-1} (y_-^u(\lambda, 0))^{\dagger}, \quad (5.28)$$

where the normalization (3.17) has been shown in the proof of Lemma 5.2, see (5.26). Therefore, due to (5.20) the Evans function (3.18) is given by

$$\mathcal{E}(\lambda) = \det(y_+(\lambda, 0)|y_-(\lambda, 0)) = \mathcal{W}(u_-, u_+), \quad (5.29)$$

while the matrices  $\mathcal{Y}_U(\lambda), \mathcal{Y}_V(\lambda)$  in Theorem 3.11 are given by

$$\mathcal{Y}_U(\lambda) = y_+(\lambda, 0)y_-(\lambda, 0)^\perp, \quad \mathcal{Y}_V(\lambda) = y_-(\lambda, 0)y_+(\lambda, 0)^\perp. \quad (5.30)$$

Furthermore, the Green's kernels (3.45) of the Green's operators (3.44) are given as follows:

$$\begin{aligned} G_+(x, \xi, \lambda) &= \frac{1}{2i\lambda^{1/2}} \begin{cases} y_+^s(\lambda, x)y_+^u(\lambda, \xi)^\perp, & 0 \leq \xi \leq x, \\ y_+^u(\lambda, x)y_+^s(\lambda, \xi)^\perp, & 0 \leq x < \xi, \end{cases} \\ G_-(x, \xi, \lambda) &= \frac{1}{2i\lambda^{1/2}} \begin{cases} y_-^u(\lambda, x)y_-^s(\lambda, \xi)^\perp, & \xi \leq x \leq 0, \\ y_-^s(\lambda, x)y_-^u(\lambda, \xi)^\perp, & x < \xi \leq 0, \end{cases} \end{aligned}$$

while the function (3.50) and the vector  $[\widehat{v}]_0 = (\mathcal{G}_-(\lambda)\widehat{v})(0-) - (\mathcal{G}_+(\lambda)\widehat{v})(0+)$  from Theorem 3.18 are computed as follows (recall that  $y_\pm^s(\lambda, \cdot) = y_\pm(\lambda, \cdot)$  by (5.20)):

$$\begin{aligned} G(\lambda, x) &= \begin{cases} y_+(\lambda, x)y_-(\lambda, 0)^\perp, & x \geq 0, \\ y_-(\lambda, x)y_+(\lambda, 0)^\perp, & x < 0, \end{cases} \\ [\widehat{v}]_0 &= \frac{1}{2i\lambda^{1/2}} \left( \int_{-\infty}^0 y_-^u(\lambda, 0)y_-(\lambda, \xi)^\perp \widehat{v}(\xi) d\xi - \int_0^\infty y_+^u(\lambda, 0)y_+(\lambda, \xi)^\perp \widehat{v}(\xi) d\xi \right). \end{aligned}$$

To give a compact formula for the singular part of the RHS of (3.51) it is convenient to introduce, for given functions  $\widehat{w}, \widehat{v} \in L^2(\mathbb{R}, \mathbb{C}^2)$ , the following  $(2 \times 1)$ ,  $(2 \times 1)$  and  $(2 \times 2)$  matrices:

$$\widehat{W}(\lambda) = \begin{pmatrix} \int_0^\infty y_+(\lambda, \xi)^\top \widehat{w}(\xi) d\xi \\ \int_{-\infty}^0 y_-(\lambda, \xi)^\top \widehat{w}(\xi) d\xi \end{pmatrix}, \quad \widehat{V}(\lambda) = \begin{pmatrix} \int_0^\infty y_+(\lambda, \xi)^\perp \widehat{v}(\xi) d\xi \\ \int_{-\infty}^0 y_-(\lambda, \xi)^\perp \widehat{v}(\xi) d\xi \end{pmatrix}, \quad (5.31)$$

$$\widehat{\mathcal{Y}}(\lambda) = \frac{1}{2i\lambda^{1/2}} \begin{pmatrix} -y_-(\lambda, 0)^\perp y_+^u(\lambda, 0) & y_-(\lambda, 0)^\perp y_-^u(\lambda, 0) \\ -y_+(\lambda, 0)^\perp y_+^u(\lambda, 0) & y_+(\lambda, 0)^\perp y_-^u(\lambda, 0) \end{pmatrix}. \quad (5.32)$$

Plugging (5.21) in (5.32), a short calculation reveals the following concretization of formulas (3.51) and (3.53) for the case of the Schrödinger operator.

**Theorem 5.3.** *Assume  $V \in L^1(\mathbb{R})$  and use linearly independent functions  $\widehat{w}_j, \widehat{v}_k \in L^2(\mathbb{R}, \mathbb{C}^2)$ ,  $j = 1, \dots, m$ ,  $k = 1, \dots, \ell$ , in (5.31). Then the singular part of the RHS of (3.51) can be expressed as follows:*

$$\frac{1}{\mathcal{E}(\lambda)} \langle \widehat{w}_j, G(\lambda, \cdot)[\widehat{v}_k]_0 \rangle_{\mathbb{R}} = \widehat{W}_j(\lambda)^\top \widehat{\mathcal{Y}}(\lambda) \widehat{V}_k(\lambda), \quad \lambda \in \Omega. \quad (5.33)$$

As in (3.53), using formula (5.33) the singular part of  $E_{jk}(\lambda)$  near the eigenvalue  $\lambda_n$ ,  $n = 1, \dots, \varkappa$ , of the operator  $H$  can be computed as follows:

$$E_{jk}^{sing}(\lambda) = \frac{\rho_n}{\lambda - \lambda_n} \left( \int_{-\infty}^\infty y_-(\lambda_n, \xi)^\top \widehat{w}_j(\xi) d\xi \right)^\top \left( \int_{-\infty}^\infty y_+(\lambda_n, \xi)^\perp \widehat{v}_k(\xi) d\xi \right). \quad (5.34)$$

We recall that at  $\lambda = \lambda_n$  the solutions  $y_+(\lambda_n, \cdot)$  and  $y_-(\lambda_n, \cdot)$  from (5.16) are proportional, see (5.8), and that the residue  $\rho_n$  of the function  $1/\mathcal{W}(\lambda)$  at the point  $\lambda_n$  is given by

$$\rho_n = \frac{1}{2\pi i} \int_{\gamma_n} \frac{d\lambda}{\mathcal{W}(u_-(\lambda, \cdot), u_+(\lambda, \cdot))} \quad (5.35)$$

for a sufficiently small circle  $\gamma_n$  centered at  $\lambda_n$ , see (5.7).

We now consider approximation (5.4) of the Schrödinger equation on the finite segment  $[x_-^N, x_+^N]$ . Let  $u_{\pm}^N$  denote the Jost solutions corresponding to the truncated potential  $V^N$  defined on  $\mathbb{R}$  by  $V^N(x) = V(x)$ ,  $x \in [x_-^N, x_+^N]$  and  $V^N(x) = 0$  otherwise, that is, the solutions of the truncated Volterra equations

$$u_{\pm}^N(\lambda, x) = e^{\pm i\lambda^{1/2}x} - \int_0^{x_{\pm}^N} \lambda^{-1/2} \sin(\lambda^{1/2}(x - \xi)) V(\xi) u_{\pm}^N(\lambda, \xi) d\xi, \quad x \in \mathbb{R}. \quad (5.36)$$

Similarly to (5.16), we denote by

$$y_{\pm}^N(\lambda, x) = \begin{pmatrix} u_{\pm}^N(\lambda, x) \\ (u_{\pm}^N)'(\lambda, x) \end{pmatrix}, \quad x \in \mathbb{R}, \quad \lambda \in \Omega, \quad (5.37)$$

the corresponding solutions of the first order differential equation (5.2), (5.3). It is easy to see from (5.36) that these solutions satisfy the following conditions:

$$y_+^N(\lambda, x_+^N) = e^{i\lambda^{1/2}x_+^N} \mathbf{v}, \quad y_-^N(\lambda, x_-^N) = e^{-i\lambda^{1/2}x_-^N} \mathbf{w}, \quad (5.38)$$

where  $\mathbf{v}, \mathbf{w}$  are the eigenvectors of  $A(\lambda, \infty)$  defined after equations (5.18), (5.19).

We will now identify the boundary conditions as required in (4.1) and (4.16). Since  $A_+(\lambda) = A_-(\lambda) = A(\lambda, \infty)$ , we have  $P_+(\lambda) = P_-(\lambda) = (2i\lambda^{1/2})^{-1} \mathbf{v} \mathbf{w}^\perp$ ,  $\mathcal{R}(P_+(\lambda)) = \text{span}\{\mathbf{v}\}$ ,  $\mathcal{N}(P_+(\lambda)) = \text{span}\{\mathbf{w}\}$ , and thus the discussion in Section 4 leading to (4.16) yields

$$R_+(\lambda) = \frac{1}{2i\lambda^{1/2}} \begin{pmatrix} 0 & 0 \\ i\lambda^{1/2} & -1 \end{pmatrix}, \quad R_-(\lambda) = \frac{1}{2i\lambda^{1/2}} \begin{pmatrix} i\lambda^{1/2} & 1 \\ 0 & 0 \end{pmatrix}. \quad (5.39)$$

In particular, due to (5.38) the solutions  $y_{\pm}^N(\lambda, \cdot)$  from (5.37) satisfy the boundary conditions

$$R_-(\lambda) y(\lambda, x_-^N) + R_+(\lambda) y(\lambda, x_+^N) = 0 \quad (5.40)$$

used in (4.1) to define the approximate operator pencil  $F_N^{(I)}(\lambda)$ .

We are ready to formulate the final convergence result of this section for the Schrödinger case. Let  $y_{\pm}^N(\lambda, \cdot)$  from (5.37) be the solutions of the equation (5.2), (5.3) satisfying (5.36), (5.38), and choose linearly independent functions  $\widehat{w}_j, \widehat{v}_k \in L^2(\mathbb{R}, \mathbb{C}^2)$ ,  $j = 1, \dots, m$ ,  $k = 1, \dots, \ell$ . Similarly to (5.34), at the eigenvalues  $\lambda_n$ ,  $n = 1, \dots, \varkappa$ , of the operator  $H$  we consider

$$E_{jk}^{N, \text{sing}}(\lambda) = \frac{\rho_n^N}{\lambda - \lambda_n} \left( \int_{-\infty}^{\infty} y_-^N(\lambda_n, \xi)^\top \widehat{w}_j(\xi) d\xi \right)^\top \left( \int_{-\infty}^{\infty} y_+^N(\lambda_n, \xi)^\perp \widehat{v}_k(\xi) d\xi \right), \quad (5.41)$$

where  $\rho_n^N$  is defined by (5.35) with the Jost solutions  $u_{\pm}(\lambda, \cdot)$  replaced by  $u_{\pm}^N(\lambda, \cdot)$ .

**Theorem 5.4.** *Assume  $V \in L^1(\mathbb{R})$ . Then the singular part (3.53) of the matrix  $E(\lambda)$  associated with the operator pencil  $F^{(I)}(\lambda)$  on the whole line is the limit as  $N \rightarrow \infty$  of the matrices  $E^{N, \text{sing}}(\lambda)$  in (5.41) associated with the operator pencil  $F_N^{(I)}(\lambda)$  on  $[x_-^N, x_+^N]$  with the boundary conditions (5.40).*

*Proof.* Since the solutions  $y_+^s(\lambda, \cdot)$  and  $y_+^u(\lambda, \cdot)$  from (5.18) are linearly independent solutions of (5.2), (5.3) and  $y_+^N(\lambda, \cdot)$  is a solution of the same equation, we may choose constants  $\alpha_+^N, \beta_+^N$  such that  $y_+^N(\lambda, x) = \alpha_+^N y_+^s(\lambda, x) + \beta_+^N y_+^u(\lambda, x)$  for all  $x \in \mathbb{R}_+$ . Solving the last equation for  $\alpha_+^N, \beta_+^N$  at  $x = x_+^N$  and using (5.38), (5.26) yields

$$\begin{aligned} \alpha_+^N &= \frac{\det(y_+^N(\lambda, x_+^N) | y_+^u(\lambda, x_+^N))}{-2i\lambda^{1/2}} = \frac{\det(\mathbf{v} | e^{i\lambda^{1/2}x_+^N} y_+^u(\lambda, x_+^N))}{-2i\lambda^{1/2}} \rightarrow 1, \\ e^{-2i\lambda^{1/2}x_+^N} \beta_+^N &= \frac{\det(e^{-i\lambda^{1/2}x_+^N} y_+^s(\lambda, x_+^N) | \mathbf{v})}{-2i\lambda^{1/2}} \rightarrow 0 \end{aligned} \quad (5.42)$$

as  $N \rightarrow \infty$  due to (5.18) (we recall that  $\operatorname{Re}(i\lambda^{1/2}) < 0$ ). A similar argument shows that if  $y_-^N(\lambda, x) = \alpha_-^N y_-^s(\lambda, x) + \beta_-^N y_-^u(\lambda, x)$  then

$$\alpha_-^N \rightarrow 1, \quad e^{2i\lambda^{1/2}x_-^N} \beta_-^N \rightarrow 0 \quad \text{as } N \rightarrow \infty. \quad (5.43)$$

Computing Wronskians and using (5.20), (5.42), (5.43) we conclude that

$$\mathcal{W}(y_-^N(\lambda, \cdot), y_+^N(\lambda, \cdot)) = \mathcal{W}(\alpha_-^N y_-^s(\lambda, \cdot) + \beta_-^N y_-^u(\lambda, \cdot), \alpha_+^N y_+^s(\lambda, \cdot) + \beta_+^N y_+^u(\lambda, \cdot))$$

converges to  $\mathcal{W}(y_-(\lambda, \cdot), y_+(\lambda, \cdot))$  as  $N \rightarrow \infty$  (and even uniformly in  $\lambda$  on compacta in  $\Omega$  because the convergence in (5.18), (5.19) is uniform on compacta). It follows that  $\rho_n^N \rightarrow \rho_n$  as  $N \rightarrow \infty$ .

It remains to show that the integral terms in (5.41) converge to the respective integral terms in (5.34). The latter fact follows from the assertions

$$\|y_\pm(\lambda, \cdot) - y_\pm^N(\lambda, \cdot)\|_{L^2(\mathbb{R}_\pm \cap [x_\pm^N, x_\pm^N])} \rightarrow 0 \quad \text{as } N \rightarrow \infty \quad (5.44)$$

using the Cauchy-Schwartz inequality. To prove (5.44) for  $\mathbb{R}_+$  (the argument for  $\mathbb{R}_-$  is similar), we recall that  $y_+^s(\lambda, \cdot) = y_+(\lambda, \cdot)$  by (5.20), and then use the representation  $y_+^N(\lambda, \cdot) = \alpha_+^N y_+(\lambda, \cdot) + \beta_+^N y_+^u(\lambda, \cdot)$  to estimate

$$\begin{aligned} \|y_+(\lambda, \cdot) - y_+^N(\lambda, \cdot)\|_{L^2([0, x_+^N])} &\leq |1 - \alpha_+^N| \|y_+(\lambda, \cdot)\|_{L^2([0, x_+^N])} \\ &\quad + |\beta_+^N| \left( \int_0^{x_+^N} e^{-2\operatorname{Re}(i\lambda^{1/2})x} \cdot e^{2\operatorname{Re}(i\lambda^{1/2})x} |y_+^u(\lambda, x)|^2 dx \right)^{1/2} \\ &\leq |1 - \alpha_+^N| \|y_+(\lambda, \cdot)\|_{L^2(\mathbb{R}_+)} + c|\beta_+^N| (e^{-2\operatorname{Re}(i\lambda^{1/2})x_+^N} - 1)^{1/2} \end{aligned}$$

since the function  $e^{i\lambda^{1/2}x} y_+^u(\lambda, x)$  is bounded on  $\mathbb{R}_+$  due to (5.18). Now (5.42) implies (5.44) finishing the proof of the theorem.  $\blacksquare$

## 6. NUMERICAL EXPERIMENTS FOR THE FITZHUGH-NAGUMO EQUATION

We apply the contour-method to investigate spectral stability of traveling waves in the FitzHugh-Nagumo system (FHN). A traveling wave in FHN is a solution of

$$\begin{aligned} 0 &= u'' + cu' + u - \frac{1}{3}u^3 - v, \\ 0 &= cv' + \Phi(u + a - bv). \end{aligned} \quad (6.1)$$

For the standard parameter values  $a = 0.7$ ,  $b = 0.8$ ,  $\Phi = 0.08$  (see [M], [BL, § 5]) one finds both, a stable pulse with speed  $c \approx -0.812$  and an unstable pulse with speed  $c \approx -0.514$ . We choose the latter one and denote it by  $(\bar{u}, \bar{v})$ .

Linearization about this pulse leads to the linear operator

$$\mathcal{L} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u'' + cu' + u - \bar{u}^2 u - v \\ cv' + \Phi u - \Phi b v \end{pmatrix}.$$

For spectral stability one has to analyze the location of the spectrum of this operator. The eigenvalue problem reads

$$(\lambda I - \mathcal{L}) \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ in } L^2(\mathbb{R}, \mathbb{C}^2).$$

From the dispersion relation one can show (e.g. [BL, § 5] or [RM, § 7]) that there is no essential spectrum in  $\Omega = \{\operatorname{Re} \lambda > -0.064\}$ . Thus, we may use the circle  $\Gamma = \{\lambda \in \mathbb{C} : |\lambda - 1| = 1.05\}$  for the contour method. Equation (2.3) becomes

$$(\lambda I - \mathcal{L}) \begin{pmatrix} y_k^1(\lambda) \\ y_k^2(\lambda) \end{pmatrix} = \hat{v}_k \text{ in } L^2(\mathbb{R}, \mathbb{C}^2). \quad (6.2)$$

For the computation we choose  $\hat{v}_k$  as a random linear combination from the  $2M$ -dimensional space

$$\tilde{\mathcal{K}}_M = \operatorname{span} \left\{ \begin{pmatrix} \varphi_j \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \varphi_j \end{pmatrix} : j = 0, \dots, M-1 \right\} \subset L^2(\mathbb{R}, \mathbb{C})^2,$$

$$\varphi_j(x) = \max\{0, 1 - |x - x_j|\}, \quad x_j = -5 + \frac{10j}{M-1}, \quad j = 0, \dots, M-1.$$

More precisely,  $\hat{v}_k = \sum_{j=0}^{M-1} (\xi_k^j(\varphi_j, 0)^\top + \zeta_k^j(0, \varphi_j)^\top)$ , with  $\xi_k^j$  and  $\zeta_k^j$  independent and normally distributed random variables.

With  $z = (y_k^1, (y_k^1)', y_k^2)^\top$  we rewrite (6.2) as a first order system (see Section 3.2),

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}' = \begin{pmatrix} z_2 \\ \lambda z_1 - cz_2 - z_1 + \bar{u}^2 z_1 + z_3 \\ -\frac{1}{c}\Phi z_1 + \frac{1}{c}(\lambda + \Phi b)z_3 \end{pmatrix} + \begin{pmatrix} 0 \\ -\hat{v}_k^1 \\ -\frac{1}{c}\hat{v}_k^2 \end{pmatrix} \text{ in } L^2(\mathbb{R}, \mathbb{C}^3). \quad (6.3)$$

For the approximation (4.1) of (6.3) on a bounded interval we choose projection and periodic boundary conditions, which both satisfy (4.4). For these boundary conditions, we showed in Theorem 4.7 exponential error estimates (with a twice as good rate for the projection boundary conditions), when the  $\hat{v}_k$  are compactly supported, a property shared by the linear combinations of hat functions  $\hat{v}_k \in \tilde{\mathcal{K}}_M$ . Furthermore, for the contour method we choose

- $l = 10$  right hand sides from  $\tilde{\mathcal{K}}_{40}$ ,
- $m = 401$  functionals  $\hat{w}_j = \delta_{x_j} \in \mathcal{H}'$ , where  $\delta_{x_j} \in \mathcal{M}_b^c$  is the Dirac measure at  $x_j = -2 + \frac{j}{100}$ ,  $j = 0, \dots, 400$ ,
- symmetric finite intervals  $J = [-\frac{L}{2}, \frac{L}{2}]$  of length  $L$ ,
- the number  $\varkappa$  for the rank test (2.15) such that the singular values of  $D_0^N$  satisfy  $\sigma_1 \geq \dots \geq \sigma_\varkappa \geq \theta \sigma_1 > \sigma_{\varkappa+1}$  for some given  $\theta > 0$ .

There are always two eigenvalues inside the circle, the zero and the unstable eigenvalue, see Figure 5 (a). We take the unstable eigenvalue for tests of accuracy. A highly accurate reference eigenfunction is computed by applying Newton's method to the discrete boundary eigenvalue problem (trapezoidal method on a large interval  $[-50, 50]$  at small step-size  $\Delta x = 0.01$ ). The same step-size  $\Delta x = 0.01$  is used when solving (4.1) on equidistant grids for the contour method.

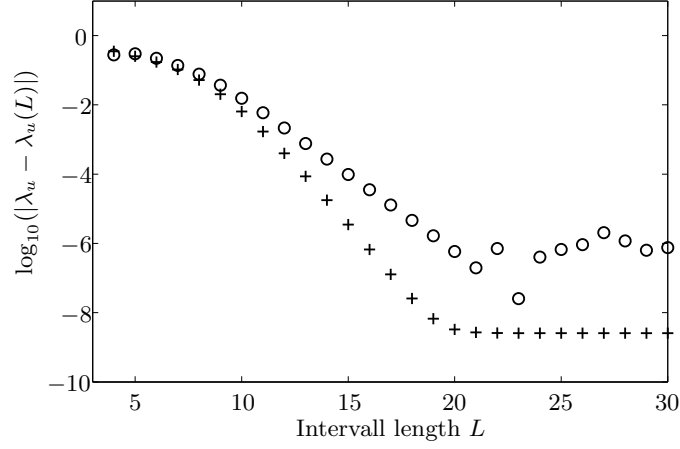


FIGURE 1. Convergence of the approximate unstable eigenvalue for projection bc's (+) and periodic bc's (o).

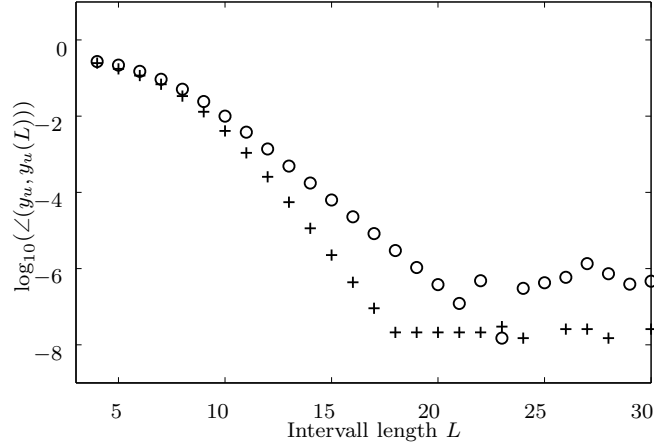


FIGURE 2. Convergence of the approximate eigenfunction for projection boundary conditions (+) and periodic bc's (o).

**6.1. Dependence on the interval size.** In our first experiment we vary the length of the interval for the finite boundary value problems (4.1). The other data is fixed. In particular, we use a large number of quadrature points ( $M = 100$ ) on the contour and  $\theta = 10^{-8}$  for determining  $\varkappa$  as above to keep the influence of the rank test small.

In Figure 1 we plot the distance of the approximate unstable eigenvalue  $\lambda_u(L)$  obtained by the contour method to the reference value  $\lambda_u$ . For the contour method (4.1) is solved on  $[-\frac{L}{2}, \frac{L}{2}]$  with periodic (o) and projection boundary conditions (+). For both boundary conditions one finds an exponential rate of convergence with a significantly better rate for the second one, as predicted by Theorem 4.7.



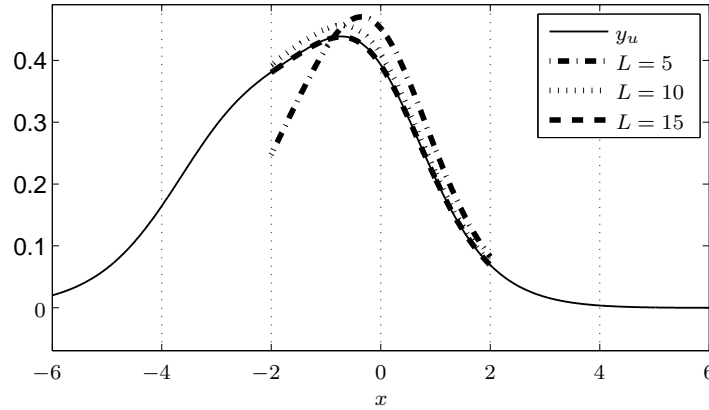


FIGURE 3. First component of  $y_u$  and its approximation by the contour method for different interval sizes.

The same observation is true for the convergence of the eigenfunction. This is shown in Figure 2, where we compare the angle between the approximate eigenfunction  $y_u(L)$  and the reference eigenfunction  $y_u$ . For the approximate eigenfunction, we use (2.21)–(2.24) with hat functions  $\hat{u}_k(x) = \max\{0, 1 - |k - 200 - 100x|\}$ ,  $k = 0, 1, \dots, 400$ , which, together with  $\hat{w}_k$ , form a biorthogonal system.

In Figure 3 we plot the  $u$ -component of the reference eigenfunction (which is actually an approximation on  $[-50, 50]$ ) and compare it to the approximate eigenfunctions obtained for the different interval lengths  $L = 5, 10, 15$  with projection bc's. The approximate eigenfunctions are only shown on  $[-2, 2]$  which is contained in all domains of definition.

**6.2. Dependence on the number of quadrature points.** In our second experiment we fix the interval to  $[-50, 50]$  with projection boundary conditions and vary the number of quadrature points. As functionals  $\hat{w}_j$  we choose the point evaluation at all grid points. We determine the rank as before and take  $\theta = 10^{-10}$  to keep the influence of the rank test small. All other data are the same as in the first experiment. The results are shown in Figure 4.

One observes exponential convergence rate with respect to the number of quadrature points, see [B] for a proof. For the eigenvalue errors there are some apparent resonances which have not yet been investigated further. It turns out that quadrature errors dominate in this case and that error plots are almost identical for periodic boundary conditions.

**6.3. Dependence on the rank test.** We now keep all data fixed but vary the rank test by prescribing the value of  $\varkappa$ .

All other data are the same as in the previous experiment except that we choose 45 quadrature points on the contour. In Table (c) from Figure 5 we list all singular values of the numerical approximation  $D_0^N$  in this case. In Figures 5 (a) and (b) we plot the approximate eigenvalues for  $\varkappa = 2$  and  $\varkappa = 10$ , respectively.

It turns out that the two eigenvalues inside the circle are nearly independent of  $\varkappa$  for  $\varkappa \geq 2$ , while the eigenvalues outside heavily depend on  $\varkappa$ .

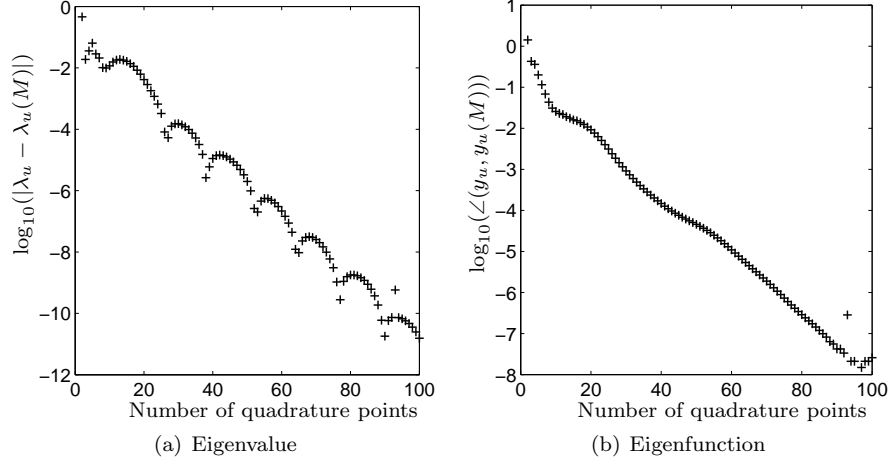


FIGURE 4. Convergence of eigenvalue and eigenfunction with increasing number of quadrature points.

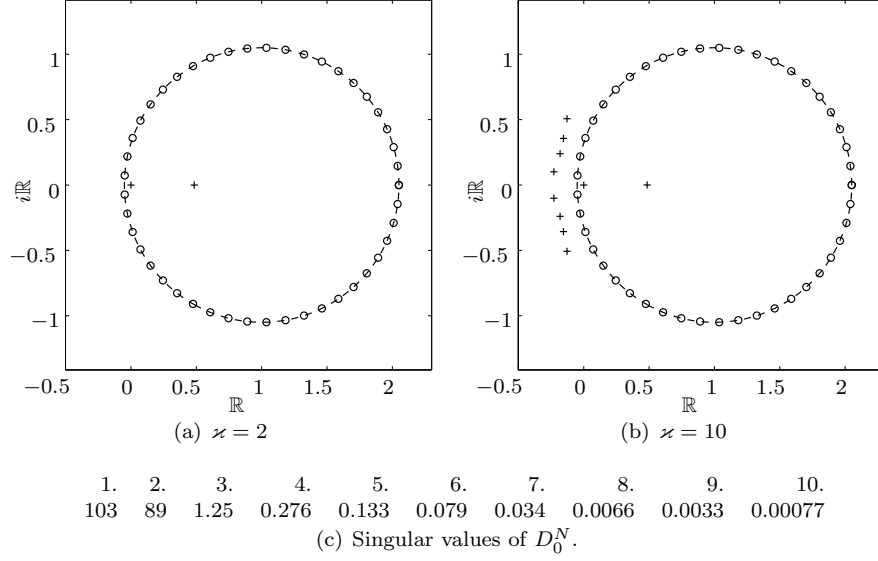


FIGURE 5. Influence of the rank test on the spectrum for two different values of  $\kappa$  (see (2.15)).  $\circ$ : Quadrature points,  $+$ : approximate eigenvalues.

It is shown in [B] that values outside but close to the contour still represent good approximations of eigenvalues. In our example, however, these eigenvalues are generated by the essential spectrum of the continuous problem which lies very close to the contour. This case is not covered by the analysis in [B] and requires further investigations.

APPENDIX A. EMBEDDING RESULTS FOR THE FUNCTION SPACES  $\mathcal{H}$  AND  $\mathcal{H}_J$ 

The aim of this appendix is to show several embedding properties for the function spaces used in Sections 3.2 and 4. Recall the Banach spaces (3.37), (4.2), i.e.

$$\mathcal{H} = \{y \in L^2(\mathbb{R}, \mathbb{C}^d) : y \in AC_{\text{loc}}, -y' + By \in L^2(\mathbb{R}, \mathbb{C}^d)\}.$$

with norm  $\|y\|_{\mathcal{H}}^2 = \|y\|_{L^2}^2 + \|-y' + By\|_{L^2}^2$  and

$$\mathcal{H}_J = \{y \in L^2(J, \mathbb{C}^d) : y \in AC(J), -y' + By \in L^2(J, \mathbb{C}^d)\},$$

with norm  $\|y\|_{\mathcal{H}_J}^2 = \|y\|_{L^2(J)}^2 + \|-y' + By\|_{L^2(J)}^2$ .

In the following we assume  $B \in L^1(\mathbb{R}, \mathbb{C}^{d,d})$  and let  $T(x, x_0)$ ,  $x, x_0 \in \mathbb{R}$ , denote the solution operator of  $-y' + B(x)y$  (that is, the propagator of the differential equation  $y' = B(x)y$  on  $\mathbb{R}$ ) in the sense of Carathéodory, i.e.  $T(\cdot, x_0)$  is a mild solution of the initial value problem  $Y' = B(x)Y$ ,  $Y(x_0) = I_d$  in the sense of Carathéodory, or equivalently

$$T(x, x_0) = I_d + \int_{x_0}^x B(\xi)T(\xi, x_0) d\xi, \quad x, x_0 \in \mathbb{R}. \quad (\text{A.1})$$

Then  $T \in C(\mathbb{R}^2, \mathbb{C}^{d,d})$  and  $T(\cdot, x_0) \in AC_{\text{loc}}(\mathbb{R})$ . Gronwall's inequality yields for all  $x, x_0 \in \mathbb{R}$ :

$$\|T(x, x_0)\| \leq \exp\left(\int_{x_0}^x \|B(\xi)\| d\xi\right) \leq \exp(\|B\|_{L^1}) =: K. \quad (\text{A.2})$$

Every  $y \in \mathcal{H}_J$  for  $J \subseteq \mathbb{R}$  satisfies the equation  $-y' + By = z$  on  $J$  for some  $z \in L^2(J, \mathbb{C}^d)$  and hence for all  $x, x_0 \in J \subseteq \mathbb{R}$  we have

$$y(x) = T(x, x_0)y(x_0) - \int_{x_0}^x T(x, \xi)z(\xi) d\xi. \quad (\text{A.3})$$

This implies for all  $x, x_0 \in J \subseteq \mathbb{R}$  the estimate

$$|y(x) - y(x_0)| \leq K\|B\|_{L^1([x, x_0])}|y(x_0)| + \sqrt{|x - x_0|}K\|y\|_{\mathcal{H}_J}, \quad (\text{A.4})$$

where we use  $[x, x_0]$  to denote the interval  $[x_0, x]$  in case  $x_0 < x$ . Indeed, (A.4) follows from

$$\|T(x, x_0) - I_d\| \leq K\|B\|_{L^1([x, x_0])}, \quad (\text{A.5})$$

and

$$\left\| \int_{x_0}^x T(x, \xi)z(\xi) d\xi \right\| \leq \sqrt{|x - x_0|}K\|z\|_{L^2}, \quad (\text{A.6})$$

which are easily obtained from (A.1) and (A.2).

Our first lemma shows asymptotic decay of elements in  $\mathcal{H}$ , that is, embedding of  $\mathcal{H}$  in the space  $C_0(\mathbb{R}, \mathbb{C}^d) = \{y \in C(\mathbb{R}, \mathbb{C}^d) : \lim_{|x| \rightarrow \infty} y(x) = 0\}$ . We supply a short proof and refer to [CL, Lem.3.16] for a more general case.

**Lemma A.1.** *Assume  $B \in L^1(\mathbb{R}, \mathbb{C}^{d,d})$ . Then  $\mathcal{H} \subset C_0(\mathbb{R}, \mathbb{C}^d)$  and  $\mathcal{H}_J \subset C(J, \mathbb{C}^d)$ .*

*Proof.* Inclusions  $\mathcal{H} \subset C(\mathbb{R}, \mathbb{C}^d)$  and  $\mathcal{H}_J \subset C(J, \mathbb{C}^d)$  follow from (A.3). Assume that for some  $y \in \mathcal{H}$  there is a sequence  $(x_N)_{N \in \mathbb{N}} \subset \mathbb{R}$  with  $|x_N| \rightarrow \infty$ , so that  $|y(x_N)| \geq \nu > 0$ . Without loss of generality we may assume  $x_{N+1} \geq x_N + 1$ . Now let  $\delta_0 = \min\{\frac{1}{2}, \nu^2(3K\|y\|_{\mathcal{H}})^{-2}\}$ . Since  $B \in L^1(\mathbb{R}, \mathbb{C}^{d,d})$  we can choose  $0 < \delta_1 \leq \delta_0$ ,

such that  $K \int_{\mathcal{M}} \|B(x)\| dx \leq \frac{1}{3}$  for all measurable  $\mathcal{M} \subset \mathbb{R}$  with  $\text{meas}(\mathcal{M}) \leq 2\delta_1$ . For  $|x - x_N| \leq \delta_1$  inequality (A.4) then implies

$$|y(x)| \geq |y(x_N)| - \frac{1}{3}|y(x_N)| - \frac{\nu}{3} \geq \frac{\nu}{3}.$$

This leads to a contradiction via the estimate

$$\|y\|_{\mathcal{H}}^2 \geq \int_{\mathbb{R}} |y(x)|^2 dx \geq \sum_{N=0}^{\infty} \int_{x_N - \delta_1}^{x_N + \delta_1} |y(x)|^2 dx = \infty.$$

■

Next, we show that embedding of  $\mathcal{H}_J$  in  $L^\infty(J, \mathbb{C}^d)$  is uniform for all  $J \subseteq \mathbb{R}$ .

**Lemma A.2.** *There exists a constant  $C > 0$  such that for all intervals  $J = [a, b] \subset \mathbb{R}$  with  $b - a \geq 1$  and for  $J = \mathbb{R}$ ,*

$$\|y\|_{L^\infty(J)} \leq C \|y\|_{\mathcal{H}_J}, \quad \text{for all } y \in \mathcal{H}_J. \quad (\text{A.7})$$

*Proof.* Let  $y \in \mathcal{H}_J$ , then  $\|y\|_{L^\infty} < \infty$  and there is  $\bar{x} \in J$  with  $|y(\bar{x})| = \|y\|_{L^\infty}$  by Lemma A.1. Let  $\delta \in (0, \min(1, K^{-2}))$  be so small that  $K \int_{\mathcal{M}} |B(x)| dx \leq \frac{1}{2}$  holds for all measurable  $\mathcal{M} \subset \mathbb{R}$  with  $\text{meas}(\mathcal{M}) \leq \delta$ . Let  $C = 2\sqrt{(2 + 2\delta)/\delta}$ . If  $\|y\|_{L^\infty}^2 \leq 8\|y\|_{\mathcal{H}_J}^2$  then (A.7) holds since  $C > \sqrt{8}$ . If  $|y(\bar{x})|^2 > 8\|y\|_{\mathcal{H}_J}^2$  then, by (A.4), for all  $x \in J$  with  $|x - \bar{x}| \leq \delta$  we have

$$|y(x)| \geq |y(\bar{x})| - \frac{1}{2}|y(\bar{x})| - \sqrt{\delta_1}K\|y\|_{\mathcal{H}_J} \geq \frac{1}{2}|y(\bar{x})| - \|y\|_{\mathcal{H}_J} > 0.$$

From this we obtain

$$\begin{aligned} \|y\|_{\mathcal{H}_J}^2 &\geq \int_J |y(x)|^2 dx \geq \int_{J \cap [\bar{x} - \delta, \bar{x} + \delta]} \left(\frac{1}{2}|y(\bar{x})| - \|y\|_{\mathcal{H}_J}\right)^2 dx \\ &\geq \delta \left( \frac{\|y\|_{L^\infty}^2}{4} - \|y\|_{L^\infty}\|y\|_{\mathcal{H}_J} + \|y\|_{\mathcal{H}_J}^2 \right) \geq \delta \left( \frac{\|y\|_{L^\infty}^2}{8} - \|y\|_{\mathcal{H}_J}^2 \right), \end{aligned}$$

and (A.7) follows. ■

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